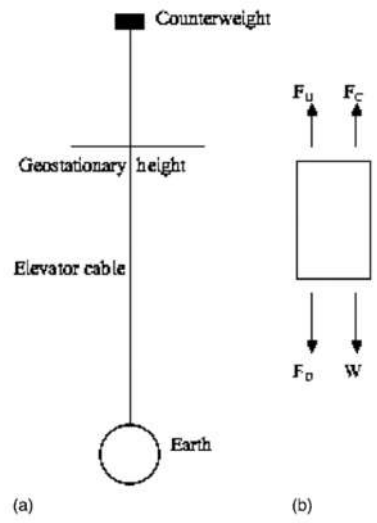


Theory Q2
Space elevator (8 points)
Solution and Marking Scheme

1 Cylindrical Space Elevator with Uniform Cross Section

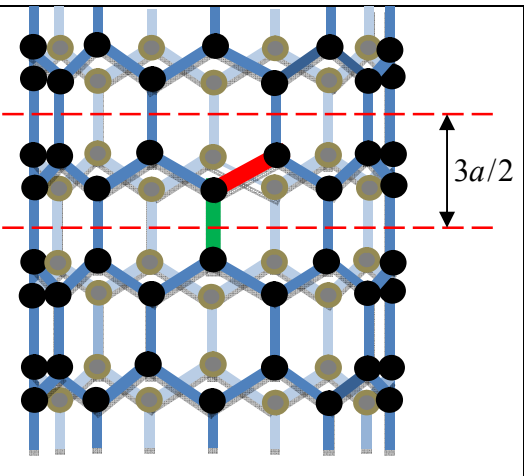
1.1 0.5pt	<p>Consider a small element of the cylinder of thickness dr at position r, there are four forces acting on that element: gravitational $\vec{W}(r)$, centrifugal $\vec{F}_C(r)$, cable tension $\vec{F}_D = \vec{T}(r)$ at position r, tension $\vec{F}_U = \vec{T}(r+dr)$ at position $r+dr$. Positive direction is chosen from the Earth center outward. The net force must be zero, therefore:</p> $-W + F_C + T(r+dr) - T(r) = 0$ $\Leftrightarrow -W + F_C + A\sigma(r+dr) - A\sigma(r) = 0'$ <p>Hence</p> $Ad\sigma = \frac{GM(A dr \rho)}{r^2} - (A dr \rho)\omega^2 r$ $\Rightarrow \frac{d\sigma}{dr} = GM\rho \left(\frac{1}{r^2} - \frac{r}{R_G^3} \right)$ <p>Note that, the tensions at the ends of the cylinder are zero. Integrating the above equation from R to R_G, one obtains the stress at R_G</p> $\sigma(R_G) = GM\rho \left[\frac{1}{R} - \frac{3}{2R_G} + \frac{R^2}{2R_G^3} \right],$ <p>Similarly, integrating from R_G to H (the distance from the Earth center to the upper end of the cylinder), one obtains the same stress at R_G</p> $\sigma(R_G) = GM\rho \left[\frac{1}{H} - \frac{3}{2R_G} + \frac{H^2}{2R_G^3} \right]$ <p>Equating the two above expressions, one arrives to the equation:</p> $RH^2 + R^2H - 2R_G^3 = 0,$ <p>from where H is determined: $H = \frac{R}{2} \left[\sqrt{1 + 8 \left(\frac{R_G}{R} \right)^3} - 1 \right] = 1.51 \times 10^5 \text{ km.}$</p> <p>The height of the cylinder $L = H - R = \frac{R}{2} \left[\sqrt{1 + 8 \left(\frac{R_G}{R} \right)^3} - 3 \right] = 1.45 \times 10^5 \text{ km.}$</p> <p><i>Note: Students can just equalize the net gravitational force and the net centrifugal force acting on the cylinder to obtain H correctly: full mark.</i></p>	<p>0.1</p> <p>0.1</p> <p>0.1</p> <p>0.1</p> <p>0.1</p>
1.2 0.5pt	The maximal stress is determined from the requirement	



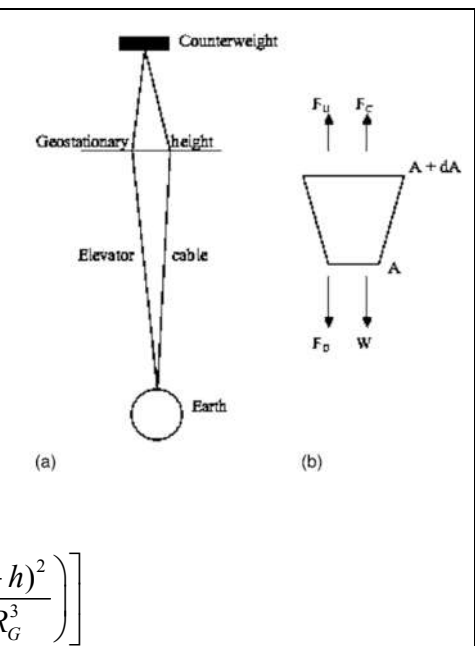
	$\frac{d\sigma}{dr} = GM\rho \left(\frac{1}{r^2} - \frac{r}{R_G^3} \right) = 0$	0.25
	which yields $r = R_G$	0.25
1.3 0.5pt	Maximal stress is expressed by	
	$\sigma(R_G) = GM\rho \left[\frac{1}{R} - \frac{3}{2R_G} + \frac{R^2}{2R_G^3} \right] \quad (1)$	
	$\sigma(R_G) = \rho g \left[R - \frac{3R^2}{2R_G} + \frac{R^4}{2R_G^3} \right] \quad (2)$	0.25
	Numerical calculation with $\rho = 7900 \text{ kg/m}^3$ one obtains the ratio:	
	$\frac{\sigma(R_G)}{5.0GA} = \frac{383 \text{ GPa}}{5.0 \text{ GPa}} = 76.5,$	0.25
	This ratio is much larger than 1, therefore steel is not suitable to build this kind of elevator.	
	<i>If eq. (2) is not obtained and other correct equation like eq. (1) is derived - 0.1pt from full mark (get only 0.15pt for maximal stress).</i>	

2 Carbon Nanotubes

2.1 0.25pt	Expand exponential function in series, and limit to the lowest power of x , one has $V = V_0 \left(-1 + \frac{4x^2}{a^2} \right)$ and gets $P = -V_0$ and	0.1
	$Q = \frac{4V_0}{a^2}.$	0.15
2.2 0.25pt	$F = -\frac{dV}{dx} = -\frac{8V_0}{a^2}x$	0.1
	then $k = \frac{8V_0}{a^2} = 313 \text{ Nm}^{-1}.$	0.15
2.3 0.5pt	Young's modulus of the carbon nanotube. Denote d the diameter of the carbon nanotube, one has $d = 27b / \pi.$	
	$E_1 = \frac{\text{stress } \sigma}{\text{strain } \varepsilon} = \frac{F/A}{x/a} = \frac{kx/A}{x/a} = \frac{ka}{A} = \frac{32V_0}{a\pi d^2}$	0.25
	$E = NE_1 = 342 \text{ GPa}$	0.25
2.4 0.5pt	$V_0 = \frac{1}{2}kx_{\max}^2 \Rightarrow x_{\max} = \sqrt{\frac{2V_0}{k}} = \frac{1}{2}a$	0.25
	$= 0.071 \text{ nm}$	0.25
2.5 0.5pt	Tensile strength of the carbon nanotube, $\sigma_0 = E \frac{x_{\max}}{a} = E/2 = 171 \text{ GPa}.$	0.5

<p>2.6 0.5pt</p>	<p>Volume $\frac{\pi d^2}{4} \times \frac{3a}{2}$ contains 18 carbon atoms, therefore the density of the carbon nanotube,</p> $\rho = \frac{2 \times 27 \times 12 \times 10^{-3}}{N_A \times \frac{\pi d^2}{4} \times \frac{3a}{2}} = 1440 \text{ kg/m}^3.$		<p>0.25 0.25</p>
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3 Tapered Space Elevator with Uniform Stress

<p>3.1 0.5pt</p>	<p>The solution to this section is analogous to that given in the previous section, however, now one has to take into account the fact that the stress σ is constant, but the cross section area A varies along the tower.</p> $\sigma dA = \frac{GM(A dr \rho)}{r^2} - (A dr \rho) \omega^2 r$ $\Rightarrow \frac{dA}{A} = \frac{\rho g R^2}{\sigma} \left(\frac{1}{r^2} - \frac{r}{R_G^3} \right) dr$ <p>where $g = GM / R^2$ is gravitational acceleration at the Earth surface. By integration one can obtain the tower cross section as:</p>		<p>0.25 0.25</p>
<p>3.2 0.5pt</p>	<p>Using the condition $A(H)=A(R)=A_s$ one arrives to the equation $RH^2 + R^2H - 2R_G^3 = 0$, which allows to determine</p> $H = \frac{R}{2} \left[\sqrt{1 + 8 \left(\frac{R_G}{R} \right)^3} - 1 \right] = 151000 \text{ km.}$		<p>0.25 0.25</p>
<p>3.3 0.5pt</p>	<p>The ratio $\frac{A_G}{A_s} = \exp \left[\frac{R}{2L_C} \left\{ \left(\frac{R}{R_G} \right)^3 - 3 \left(\frac{R}{R_G} \right) + 2 \right\} \right] = 1.623$ where $L_C = \frac{\sigma}{\rho g}$</p>		<p>0.5</p>
<p>3.4 1.0pt</p>	<p>Net force exerted on the counterweight must be zero</p> $\frac{GMm_c}{[R_G + h_c]^2} + A(R_G + h_c) \cdot \sigma = m_c \omega^2 [R_G + h_c],$ <p>replacing $A(R_G + h)$ from the equation for cross section area, one can determine the counterweight mass.</p>		<p>0.5</p>

$m_C = \frac{\rho A_S L_C \exp \left[\frac{R^2}{2L_C R_G^3} \left(\frac{2R_G^3 + R^3}{R} - \frac{2R_G^3 + (R_G + h_C)^3}{R_G + h_C} \right) \right]}{\frac{R^2 (R_G + h_C)}{R_G^3} \left[1 - \left(\frac{R_G}{R_G + h_C} \right)^3 \right]}$	0.50
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4 Applications

4.1	An object can leave the Earth if its energy at the distance r satisfies	
0.5pt	$E = \frac{m(\omega r)^2}{2} - \frac{GMm}{r} \geq 0$ from which $r_C = (2GM / \omega^2)^{\frac{1}{3}} = 53200km$	0.25
	In order to launch an object, the upper end of the tower must locate above the distance r_C .	0.25

4.2	<p>We denote the Earth orbital velocity as v_E, the spacecraft velocity when it's released from the tower top as $v_1 = \omega h_0$. The spacecraft can reach the furthest distance from the Sun if \vec{v}_1 is parallel to \vec{v}_E. The spacecraft velocity relative to the Sun is $v_E + v_1$. The Earth orbital radius R_E also is the smallest distance from the sun (if one neglects the tower length compared to the radius of the Earth's orbit). r_2 is the apogee distance of the spacecraft from the Sun, v_2 is its velocity at apogee. Angular momentum and energy conervation laws read</p>	
1.0pt	$m(v_E + v_1)R_E = mv_2 r_2$	0.1
	$\frac{1}{2}m(v_E + v_1)^2 - \frac{GM_S m}{R_E} = \frac{1}{2}mv_2^2 - \frac{GM_S m}{r_2}$	0.1
	<p>Here the energy term $-\frac{GMm}{h_0}$ due the earth's gravity is neglected. Eliminating v_2 one has</p>	
	$\left[(v_E + \omega h_0)^2 - \frac{2GM_S}{R_E} \right] r_2^2 + 2GM_S r_2 - (v_E + \omega h_0)^2 R_E^2 = 0$	0.1
	<p>from which $r_{Max} = r_2 = \frac{(v_E + \omega h_0)^2 R_E^2}{2GM_S - (v_E + \omega h_0)^2 R_E}$.</p>	0.1
	<p>Numerical calculation gives $r_2=5.3AU$, that covers Jupiter's orbit. Similarly, for the spacecraft to approach as close as possible to the Sun, the released velocity \vec{v}_1 must be antiparallel to \vec{v}_E. The spacecraft velocity relative to the Sun is $v_E - v_1$, r_2 is the perigee distance of the spacecraft from the Sun, v_2 is its velocity at perigee.</p>	
	<p>The previous angular momentum and energy conervation laws still hold,</p>	
	$m(v_E - v_1)R_E = mv_2 r_2$	0.1

	$\frac{1}{2}m(v_E - v_1)^2 - \frac{GM_S m}{R_E} = \frac{1}{2}mv_2^2 - \frac{GM_S m}{r_2}$	0.1
	<p>Here the energy term $-\frac{GMm}{h_0}$ due the earth's gravity is neglected. Eliminating v_2 one has</p>	
	$\left[(v_E - \omega h_0)^2 - \frac{2GM_S}{R_E} \right] r_2^2 + 2GM_S r_2 - (v_E - \omega h_0)^2 R_E^2 = 0$	0.1
	<p>from which $r_{\min} = r_2 = \frac{(v_E - \omega h_0)^2 R_E^2}{2GM_S - (v_E - \omega h_0)^2 R_E}$.</p>	0.1
	<p>Numerical calculation gives $r_{\min} = 0.43$ AU, meaning the Mercury's orbit is within our reach.</p>	0.1

References

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