Theory Q3

Thermoelectric effects and theirapplication in thermoelectric generator and refrigerator(10 points)

Solution and Marking scheme

A. Heat transfer and thermoelectric generator

A1. Heat transfer in a homogeneous conducting bar

A1.1 **0.75 pt**	Consider heat transfer in the segment dx of the bar in the steady state. Equation for the balance of the energy exchange through the cross-sectional area is written as $-kS\frac{dT(x)}{dx}+\rho\frac{dx}{S}I^2=-kS\frac{dT(x+dx)}{dx}=-kS\frac{dT(x)}{dx}-kS\frac{d^2T(x)}{dx^2}.dx$ Hence $-kS\frac{d^2T(x)}{dx^2}=\frac{\rho I^2}{S}$ (A1)	**0.25**
	Integration of (A1) gives $\frac{dT(x)}{dx}=-\frac{\rho I^2}{kS^2}x+C_1,$ (A2) $T(x)=-\frac{\rho I^2}{2kS^2}x^2+C_1x+C_2.$ (A3) Constants $C_1,\ C_2$ are derived from the boundary conditions $x=0\Rightarrow T=T_1\Rightarrow C_2=T_1,$ (A4) $x=L\Rightarrow T=T_2\Rightarrow C_1=\frac{T_2-T_1}{L}+\frac{1}{2}\frac{\rho L}{S^2k}I^2.$ (A5)	**0.25**
	Equation for the temperature distribution in the bar is $T(x)=T_1+\left(\frac{\rho LI^2}{2kS^2}-\frac{T_1-T_2}{L}\right)x-\frac{\rho I^2}{2kS^2}x^2.$ (A6)	**0.25**

A1.2 **1.0 pt**	Using (A2) –(A5) we obtain the equation for the heat current at x $q(x)=-kS\frac{dT(x)}{dx}=\frac{kS}{L}(T_1-T_2)+\frac{\rho I^2}{S}\left(x-\frac{L}{2}\right),$ (A7)	**0.5**
	at $x=0,$ and $x=L$ $q(x=0)=\frac{kS}{L}(T_1-T_2)-\frac{\rho LI^2}{2S}=K(T_1-T_2)-\frac{RI^2}{2},$ (A8)	**0.25**
	$q(x=L)=\frac{kS}{L}(T_1-T_2)+\frac{\rho LI^2}{2S}=K(T_1-T_2)+\frac{RI^2}{2}.$ (A9) Here $K=\frac{kS}{L},\ R=\frac{\rho L}{S}.$	**0.25**

A2. Relation between Peltier and Seebeck Coefficients

Thermocouple consists of two subsystems: a) the conducting electron gas that performs an ideal themodynamic cycle; b) Nuclei and bounded electrons of the bar crystal that oscillate around equillibrium positions at finite temperature and participate in heat conduction process. If the resistance of the thermocouple is neglected, these two subsystems may be considered as noninteracting, the electron gas exchanges heat only with the heat source at T_1 and the heat sink at T_2, performing the ideal Carnot cycle.

A2.1 **0.25 pt**	Electron gas receives heat from heat source due to the Peltier effect $q_1 = \pi_1 I$ (A10)	**0.25**
A2.2. **0.25 pt**	The heat amount transferred to the heat sink due to the Peltier effect $q_2 = \pi_2 I$ (A11)	**0.25**
A2.3. **0.5 pt**	Power delivered by the electron gas due to the Seebeck emf is $P = \varepsilon I = \alpha(T_1 - T_2)I$ (A12)	**0.5**
A2.4 **0.5 pt**	The efficiency of the ideal Carnot cycle applied to the thermocouple can be written as $\eta = \frac{P}{q_1}, \eta = \frac{T_1 - T_2}{T_1}$. (A13) Thus $\frac{T_1 - T_2}{T_1} = \frac{\alpha(T_1 - T_2)}{\pi_1}$ (A14) Comparing these equations, one has $\pi_1 = \alpha T_1$. This is the Peltier coefficient at the first junction contacting with the heat source. Generally, one has $\pi = \alpha T$.	**0.25** **0.25**

A3. Thermoelectric generator

A.3.1. **0.5 pt**	Power received by the thermocouple from the heat source (see also (A8)) is $q_1 = K(T_1 - T_2) + \alpha T_1 I - \frac{1}{2} I^2 R.$ (A15) Here α is the Seebeck coefficient of the thermocouple and $K = K_A + K_B = \frac{k_A S_A}{L} + \frac{k_B S_B}{L}$, (A16) $R = R_A + R_B = \frac{\rho_A L}{S_A} + \frac{\rho_B L}{S_B}$, (A17) are its thermal conductance and internal resistance. The heat sink receives a power (see also (A9)) $q_2 = K(T_1 - T_2) + \alpha T_2 I + \frac{1}{2} I^2 R.$ (A.18)	**0.25** **0.25**
A3.2. **0.75 pt**	The efficiency of the thermoelectric generator is $\eta = \frac{P_L}{q_1} = \frac{I^2 R_L}{K(T_1 - T_2) + \alpha T_1 I - I^2 R/2} = \frac{m}{\frac{K(T_1 - T_2)}{I^2 R} + \frac{\alpha T_1}{IR} - \frac{1}{2}}$. (A19) Here we use $R_L = mR$. The electrical current in the circuit is	**0.25**

	$I = \dfrac{\alpha\left(T_1 - T_2\right)}{R_L + R} = \dfrac{\alpha\left(T_1 - T_2\right)}{\left(1+m\right)R}.$ (A20) Substituting (A20) into (A19) we obtain the expession for the efficiency $\eta = \dfrac{m\left(T_1 - T_2\right)}{\dfrac{KR\left(1+m\right)^2}{\alpha^2} + T_1\left(1+m\right) - \dfrac{T_1 - T_2}{2}}.$ (A21)	0.25 0.25
A3.3. 0.25	Replacing the figure of merit $Z = \dfrac{\alpha^2}{KR}$ (A22) and $\eta_c = \dfrac{T_1 - T_2}{T_1}$ the efficiency of the ideal Carnot cycle in (A21), one has $\eta = \eta_c \dfrac{m}{\dfrac{\left(1+m\right)^2}{ZT_1} + \left(1+m\right) - \dfrac{1}{2}\eta_c}.$ (A23) From (A23) one sees that larger Z leads to the larger efficiency of the corresponding thermoelectric generator. The condition $ZT_1 \geq 1$ can be used for material application in thermoelectric generators.	 0.25

A4. The maximum efficiency

A4.1 0.25 pt	When $R_L = R$ or $m=1$, the power consumed on the load is maximum. The efficiency in that case is $\eta_P = \dfrac{T_1 - T_1}{\left[\dfrac{4}{Z} + \dfrac{3T_1 + T_2}{2}\right]}.$ (A24)	0.25
A4.2. 0.75 pt	Equation (A23) may be rewritten as $\eta = \dfrac{m}{a\left(1+m\right)^2 + b\left(1+m\right) - 1/2},$ (A25) where $a = \dfrac{1}{Z\left(T_1 - T_2\right)}$, $b = \dfrac{T_1}{T_1 - T_2}$. Equation $\dfrac{d\eta}{dm} = 0$ has the solution $M = \sqrt{1 + \dfrac{2b-1}{2a}}$ or $M = \sqrt{1 + Z\dfrac{\left(T_1 + T_2\right)}{2}}.$ (A26)	0.25 0.25 0.25
A4.3. 0.25 pt	Using (A25), (A26) we obtain the maximum efficiency of the thermoelectric generator $\eta_{\max} = \dfrac{T_1 - T_2}{T_1} \dfrac{\left(M-1\right)}{\left(M + \dfrac{T_2}{T_1}\right)}$ (A27) (*Correct expression containing either M, Z or both is also accepted*)	0.25

A5. The maximum figure of merit

A5.1 **0.5**	According to (A22) Z takes the maximum value $Z = Z_m$ when $KR = y$ is smallest. Denoting $\left(k_A S_A + k_B S_B\right)\left(\frac{\rho_A}{S_A} + \frac{\rho_B}{S_B}\right) = y,\ x = \frac{S_A}{S_B}$ one has the equation $\left(k_A x + k_B\right)\left(\frac{\rho_A}{x} + \rho_B\right) = y$. It is easily to show the function y has the minimum at x=x_m, where $x_m = \sqrt{\frac{\rho_A k_B}{\rho_B k_A}}$ or $\frac{S_A}{S_B} = \left(\frac{\rho_A k_B}{\rho_B k_A}\right)^{1/2}$. (A28)	**0.25** **0.25**
A5.2 **0.25 pt**	If the ratio of cross-sectional areas satisfies (A28) then $y_m = \left[\left(\rho_A k_A\right)^{1/2} + \left(\rho_B k_B\right)^{1/2}\right]^2$ and the maximum figure of merit of the thermocouple is $Z_m = \frac{\alpha^2}{\left[\left(\rho_A k_A\right)^{1/2} + \left(\rho_B k_B\right)^{1/2}\right]^2}$. (A.29)	**0.25**

A6. The optimal efficiency

A6.1. **0.5 pt**	The thermocouple with two bars made from material A and B has the following the figure of merit $Z_m = \frac{\alpha^2}{\left[\left(\rho_A k_A\right)^{1/2} + \left(\rho_B k_B\right)^{1/2}\right]^2} = \frac{\alpha^2}{4\rho_A k_A} = 3.15\times10^{-3}\,\mathrm{K}^{-1}$. (A.30) The optimal efficiency of the thermocouple AB when T_1= 423K, T_2 = 303K has the following value $\eta_{opt} = \frac{T_1 - T_2}{4Z_m^{-1} + \frac{3T_1 + T_2}{2}} = \frac{120}{4\frac{1}{3.2\times10^{-3}} + \frac{3\times423+303}{2}} = 5.84\%$. (A.31) The corresponding ideal Carnot efficiency for that case is $\eta_C = \frac{T_1 - T_2}{T_1} = \frac{120}{423} = 28.4\%$ (A32) $\eta_{opt} / \eta_C = 0.21$.	**0.15** **0.25** **0.1**
A6.2 **0.25 pt**	The maximum efficiency of the thermoelectric generator designed from AB materials is $M = \sqrt{1 + Z_m \frac{\left(T_1 + T_2\right)}{2}} = \sqrt{1 + 3.2\times10^{-3}\times363} = 1.46$ $\eta_{\max} = \eta_C \frac{\left(M-1\right)}{\left(M + \frac{T_2}{T_1}\right)} = 6.0\%$. (A.33)	**0.25**

B. Thermoelectric refrigerator

B1. The cooling power and the maximum temperature difference

B1.1 **0.25pt**	For cooling purpose we choose the current direction so that heat is absorbed at upper junction (temperature T_1) due to Peltier effect and transferred to the A & B bars. Using (A.9) one gets cooling power taken out from heat source at T_1 $q_C = \alpha T_1 I + K(T_1 - T_2) - \frac{RI^2}{2}$ (B.1) where K, R are thermal conductance and internal resistance of thermocouple.	**0.25**
B1.2. **0.5**	Condition for the maximum cooling power q_{CM} is founded from $\frac{dq_C}{dI} = 0$, one has $I_q = \frac{\alpha T_1}{R},$ (B2) $q_{CM} = \frac{\alpha^2 T_1}{2R} - K(T_2 - T_1).$ (B3) The maximum temperature depression is derived from the condition $q_{CM} = 0$, which gives $\Delta T_{\max} = T_2 - T_{1\min} = \frac{\alpha^2 T_{1\min}^2}{2KR} = \frac{ZT_{1\min}^2}{2}.$ (B4) Here $Z = \frac{\alpha^2}{KR}$ is the figure of merit of the thermocouple.	**0.25** **0.25**

B2. The working current

B2.1 **0.25pt**	Thermocouple AB with $Z_m = 3.15\times10^{-3}\mathrm{K}^{-1}$ is used for a refrigerator. The lowest cooling temperature T_{1min} is found from the same equation (B4) $0 = T_{1\min}^2 + \frac{2}{Z_m}T_{1\min} - \frac{2}{Z_m}T_2$ $T_{1\min} = \frac{1}{Z_m}\left(\sqrt{1+2Z_m T_2} - 1\right).$ (B5) Putting $T_2 = 300\mathrm{K}$ and $Z_m = 3.15\times10^{-3}\mathrm{K}^{-1}$ in (B.5) we obtain $T_{1\min} = 2.22\times10^2\mathrm{K}.$ (B.6)	**0.1** **0.15**
B2.2. **0.5**	Putting the value of the internal resistance $R = \frac{\rho_A L}{S_A} + \frac{\rho_B L}{S_B} = \frac{2\rho_B L}{S_B} = 4.0\times10^{-3}\Omega$ in (B2), one gets the working current $I_W = \frac{\alpha T_{1\min}}{R} = \frac{4.2\times10^{-4}\times221.5}{4\times10^{-3}}\mathrm{A} = 23.3\mathrm{A}$ (B7)	**0.25** **0.25**

B3. The coefficient of performance

B3.1 **0.5pt**	According to the energy conservation law, the power supplied by the electrical source P equals to the Joule heat plus Peltier's heat taken away in thermocouple per unit of time: $P=\alpha(T_2-T_1)I+RI^2$. (B.8) The equation for Coefficient of Performance (COP) is $$\beta=\frac{q_C}{P}=\frac{\alpha T_1 I-K\left(T_2-T_1\right)-\frac{RI^2}{2}}{\alpha(T_2-T_1)I+RI^2}$$ (B9)	**0.25** **0.25**
B3.2. **0.25**	Electrical current I_β corresponds to the maximum of the COP is found from the equation $\frac{d\beta}{dI}=0$. (B9) may be rewritten in convenience form $$\beta=-\frac{1}{2}+\frac{\alpha\left(T_1+T_2\right)I-2K\left(T_2-T_1\right)}{2\left[\alpha(T_2-T_1)+RI\right]I}$$. (B10) The equation $\frac{d\beta}{dI}=0$ leads to $-\alpha R\left(T_1+T_2\right)I^2+4K\left(T_2-T_1\right)RI+2K\alpha\left(T_2-T_1\right)^2=0$, $I^2-\frac{2K\left(T_2-T_1\right)I}{\alpha T_M}-\frac{K}{RT_M}\left(T_2-T_1\right)^2=0$, (B.11) with $\mathrm{T}_M=\frac{\left(T_2+T_1\right)}{2}$. (B.12) Solution of (B.11) is $I_\beta=\frac{K\left(T_2-T_1\right)}{\alpha T_M}\left\{\sqrt{1+Z\,\mathrm{T}_M}+1\right\}$. (B.13) (*Taking into account that* $Z=\frac{\alpha^2}{KR}$, *(B.13) can be written in other form* $I_\beta=\frac{\alpha\left(T_2-T_1\right)}{R\left\{\sqrt{1+Z\,\mathrm{T}_M}-1\right\}}$*)* (B.14)	 **0.25**
B3.3. **0.25**	Substituting (B.14) into (B.9) one has $$\beta_{\max}=\frac{T_1\left[\sqrt{1+ZT_M}-T_2/T_1\right]}{\left(T_2-T_1\right)\left[\sqrt{1+ZT_M}+1\right]}.$$ (B.15)	**0.25**