RF reflectometry for spin readout for silicon quantum computing

Introduction

Developing the idea of quantum computing into a practical technology is one of the largest outstanding challenges in science and technology. A promising path is to manipulate individual electrons in silicon transistors by time-dependent electromagnetic fields.

In this question, we investigate the use of radio frequency (RF) reflectometry and single-electron transistors to read out the state of quantum bits in silicon-based quantum computer prototypes.

Part A and Part B discuss radio wave transmission through cables and transmission lines, part C is devoted to conditions for wave reflection, part D introduces the single-electron transistor, and parts E and F introduce and ask you to optimise the method of reflectometry.

Part A: Lumped element model of a co-axial transmission line (2.0 points)

When modelling DC or low frequency signals, one often assumes that a voltage pulse travels instantaneously throughout the circuit. This assumption is valid when the wavelength of such signals is much longer than the size of the circuit, however when working with radio frequency signals, the dynamics are more complex, and we need to account for the intrinsic capacitance and inductance of our cables in our model. We model a co-axial transmission line which acts as a waveguide as described below, ignoring the small resistance of the copper and the small conductance through the dielectric. Throughout the problem, we consider the large-wavelength limit of electromagnetic waves in the co-axial cable such that electric and magnetic fields are perpendicular to the axis of the cable everywhere (the so-called transverse electromagnetic mode).

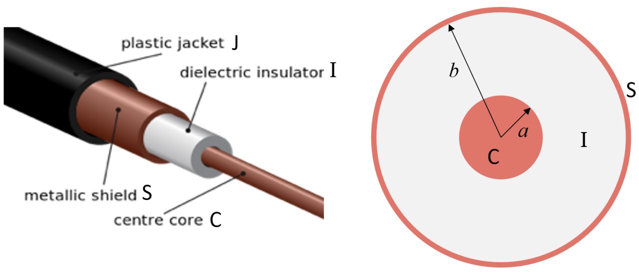


Diagram of a coaxial cable showing C - the centre core, I - the dielectric insulator, S - the metallic shield and J - the plastic jacket.

Consider a co-axial cable consisting of a copper inner core of negligible resistance, negligible magnetic permeability and radius a, covered by an outer co-axial copper shield with inner radius b. A dielectric of dimensionless relative permittivity ε_r and dimensionless relative permeability μ_r separates the layers. When electromagnetic signals propagate through the co-axial cable, they are confined between the inner core and outer shielding.

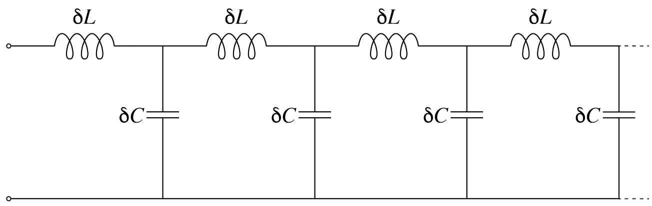
A.1	At what speed do electromagnetic waves propagate in the co-axial cable?	0.2pt

A.2	If there is a charge Δq on a length Δx of the inner core of the co-axial cable, and the outer shield is grounded, find the electric field in the region between the inner core and the shield.	0.2pt

A.3 Find the capacitance per unit length, C_x, of the co-axial cable. You may wish to consider a length Δx of the cable. 0.3pt

A.4 Find the inductance per unit length, L_x, of the cable. 0.3pt

A *lumped element* model of the cable is constructed by considering the inductance and capacitance of short sections of the cable. The inductance is assumed to be a property of the inner core, and the capacitance links the core with the shielding. A diagram of the lumped element model is shown below.



Circuit diagram of lumped element model of coaxial cable.

A.5 i. Show that the impedance Z_0 of a semi-infinite length of cable is $Z_0 = \sqrt{L_x/C_x}$.
ii. Find b/a if the cable has impedance $Z_0 = 50\ \Omega$ and is made using a dielectric material with $\varepsilon_\mathrm{r} = 4.0$ and $\mu_\mathrm{r} = 1.0$. 1.0pt

Part B: Hypothetical transmission line with return along a grounded plane (1.0 points)

An alternative hypothetical transmission line is shown in the diagram below. The input signal is sent through a very thin conductor of radius a, which is a distance $d \gg a$ from a highly conductive grounded plane. The material surrounding the conductor has dimensionless relative permittivity ε_r and dimensionless relative permeability μ_r. The return current flows along the grounded plane.

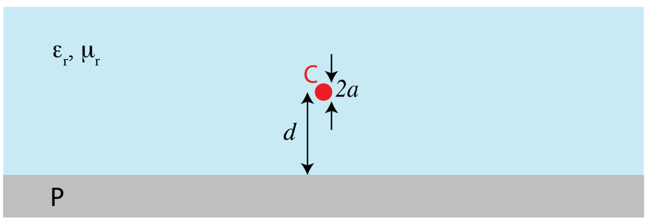


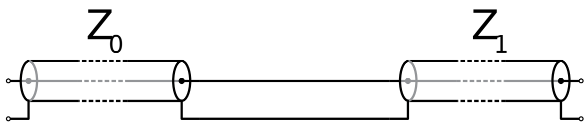
Diagram of a hypothetical transmission line showing C - the conductor of radius a, at a distance $d \gg a$ from P - the grounded conducting plane. The conductor is embedded in a material with dimensionless relative permeability ε_r and dimensionless relative permittivity μ_r.

B.1 Find an expression for the characteristic impedance of this hypothetical transmission line. 1.0pt

Part C: Basics of RF reflectometry (1.2 points)

An electromagnetic wave can propagate in a transmission line in two opposite directions. For each direction of propagation, the characteristic impedance Z_0can be used to relate the voltage V_0 and current I_0amplitudes as in the Ohm's law, $Z_0 = V_0/I_0$.

Consider an interface between two transmission lines, with characteristic impedances Z_0 and Z_1. A schematic diagram of the circuit is shown below.



Circuit diagram of a transmission line of impedance Z_0 connected to a transmission line of impedance Z_1. The physical size of the interface is much smaller than the wavelength.

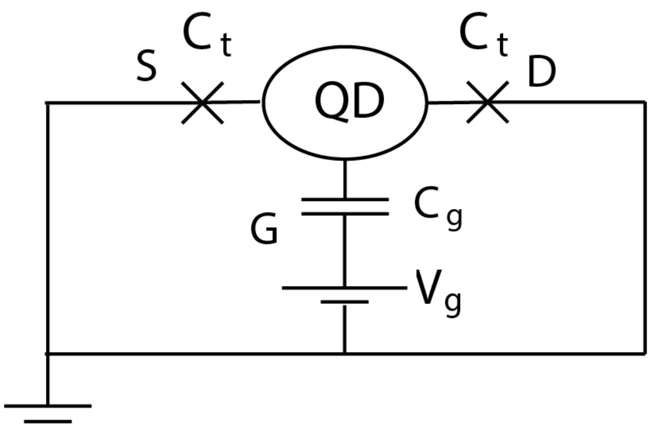
When a signal V_i sent into the transmission line with impedance Z_0reaches the interface it is partially transmitted into the second transmission line, resulting in a signal V_t in that line which propagates forward. Some of the signal may also be reflected, resulting in a backward propagating signal in the initial transmission line V_r.

C.1	Find the reflectance of the interface $\Gamma = V_\mathrm{r}/V_\mathrm{i}$.	1.0pt

C.2	State the condition(s) for the signal V_i to have gained a π phase change on reflection.	0.2pt

Part D: The single electron transistor (3.3 points)

A single electron transistor (SET) consists of a quantum dot, which is a small isolated conductor where electrons can be localised, and of several electrodes in its vicinity. The gate electrode couples capacitatively to the quantum dot, while the two other electrodes --- the source and the drain --- are connected via tunnel junctions, through which electrons can tunnel due to quantum mechanics. A simplified circuit diagram for an SET is shown in the figure.



Circuit diagram representation of an SET. QD is the quantum dot, S is the source, D is the drain and G is the gate.

The capacitance of the gate is C_g and the capacitance of the tunnel junctions is $C_t \ll C_g$. Consider C_g to be the total capacitance of the quantum dot. In this part of the problem, the source and the drain are held at zero potential, and the voltage on the gate electrode is fixed at V_g.

D.1	Consider a state of the SET in which the quantum dot contains n electrons. i. Find the electrical potential φ_n on the QD. ii. Find the amount of energy ΔE_n that is necessary to bring an additional electron from the source or the drain onto the QD.	1.5pt

If $\Delta E_n < 0$ then electrons will spontaneously tunnel into the quantum dot until such a number $\mathcal{N} > n$ is reached that $\Delta E_{\mathcal{N}} \geq 0$. The equilibrium number of electrons $\mathcal{N}$ and the corresponding addition energy $\Delta E_{\mathcal{N}}$ can be controlled by choosing the appropriate voltage V_g.

D.2	Find an expression for the maximal possible value $E_c = \max \Delta E_{\mathcal{N}}(V_g)$ of the equilibrium addition energy that can be achieved by tuning the gate voltage of the SET.	0.5pt

If $\Delta E_{\mathcal{N}} = 0$ then tunnelling of electrons does not require extra energy and SET is in a highly conductive ON state. If $\Delta E_{\mathcal{N}} > 0$, then the conductance of the SET is reduced (high-resistance OFF).

For the number of electrons on the quantum dot to remain well-defined, certain conditions need to be satisfied. Firstly, if electrons in the source or drain have thermal energies sufficient to move spontaneously onto the quantum dot, the contrast between the ON and OFF states will disappear.

D.3	Find a condition on the temperature of the electrons so that electrons cannot move onto the quantum dot by thermal excitation.	0.5pt

Secondly, tunnelling of electrons onto or off the dot limits the lifetime of their energy states. This tunnelling can be modelled using an effective resistance of the tunnel junction with the characteristic tunnelling time equal to the characteristic time for charging or discharging the quantum dot through the junction.

D.4	i. Estimate the tunnelling time for a quantum dot in terms of capacitance C_t and effective resistance R_t of the tunnel junction. ii. Find a condition on the effective resistance R_t so that the electrons in the quantum dot retain sufficiently well-defined energy for the ON and OFF states to remain distinct.	0.8pt

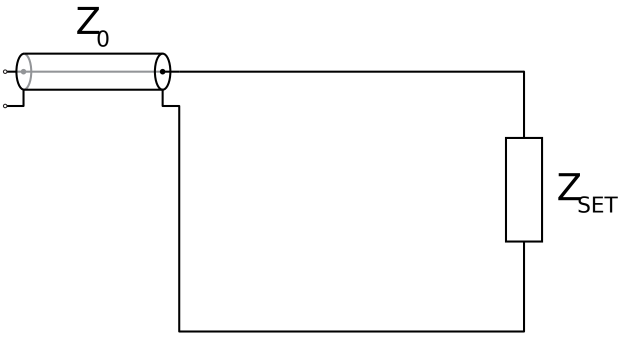
Part E: RF reflectometry to read out SET state (1.0 points)

The state of the SET is sensitive to electrical potentials created by nearby elements of the quantum circuit (such as quantum bits), and distinguishing between ON and OFF states provides a way to read out the information produced by the quantum computer. The SET in the ON state can be modelled by a resistance $R_{\mathrm{ON}} = 100\,\mathrm{k\Omega}$ while in the OFF state we can assume the SET to be a complete insulator (neglecting any capacitative connection between the source and the drain via the SET). While it is possible to determine the state of the SET by measuring the response to an input signal through the source, it is faster to do so using RF reflectometry to measure both the amplitude and phase of the reflected signal, i.e. determined the reflectance Γ.

The change in reflectance due to switching of an SET between ON and OFF states is

$$\Delta\Gamma = |\Gamma_{\mathrm{ON}} - \Gamma_{\mathrm{OFF}}| \quad , \tag{1}$$

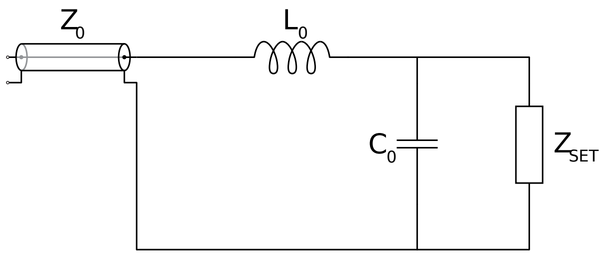
where Γ_{ON} and Γ_{OFF} are the reflectances in two different states.



Circuit diagram of transmission cable of impedance Z_0 connected to an SET.

E.1	Find the change in reflectance $\Delta\Gamma$ between the conductive and insulating states for a typical SET connected to a co-axial cable with impedance of $50\ \Omega$.	0.2pt

In order to increase the change in reflectance, and hence the sensitivity of the RF reflectometry, the circuit is modified by inclusion of an inductor. The intrinsic capacitance due to the device geometry $C_0 \approx 0.4\ \mathrm{pF}$ is also taken into account. The RF reflectometry is conducted using a signal of angular frequency ω_{rf}.



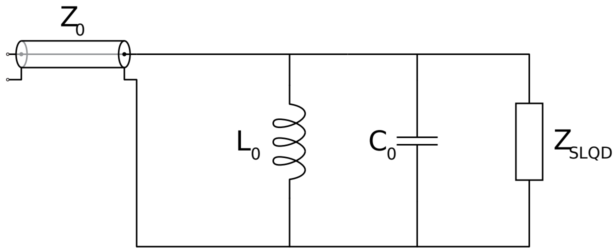
Modified SET circuit.

E.2	Estimate the value of the inductance L_0 that can result in the change in reflection on the order of one. Calculate your estimate for L_0 numerically for $\omega_{\mathrm{rf}}/(2\pi) = 100\ \mathrm{MHz}$ and compute the corresponding $\Delta\Gamma$.	0.8pt

Part F: Charge sensing with a single lead quantum dot (1.5 points)

For a scalable quantum computing architecture, the number of wires reaching each individual quantum bit need to be minimized. A promising alternative to an SET for charge sensing in silicon quantum computing is a Single Lead Quantum Dot (SLQD). In many ways it is similar to an SET, but does not have the source and drain leads. The gate is the only electrode, through which the electron energy states of the quantum dot are controlled and also through which RF reflectometry is conducted.

Like an SET, a SLQD has an OFF in which the SLQD behaves as a total insulator. In contrast to an SET, the ON state of the SLQD is capacitive, with capacitance C_{q}. In order to maximize the difference in reflectance $\Delta\Gamma$ of the SLQD, the following circuit is constructed. The parasitic capacitance $C_0 \approx 0.4\,\mathrm{pF}$ is fixed by circuit geometry, but the value of L_0 and the operating frequency can be changed to optimize the performance. The characteristic impedance of the transmission line is $Z_0 = 50\,\Omega$.



Circuit diagram of the SLQD readout circuit connected to the transmission line.

F.1	Suggest ω_{rf} and $Z_{\mathrm{C}} = \sqrt{L_0/C_0}$ that allow $\Delta\Gamma \sim 1$ for given C_0 and C_q.	1.0pt

Optimal values of L_0 are relatively large and not always technically feasible. Hence, other types of circuit elements may be needed to improve sensitivity of the reflectometry readout circuit.

F.2	Assume that L_0 (and hence Z_C) is fixed. Draw a circuit diagram showing where to place an additional element in the SLQD readout circuit and specify the parameter(s) of this element such that $\Delta\Gamma \sim 1$ can still be achieved without requiring a large inductance.	0.5pt

Theory Q1: Solutions
RF Reflectometry

Version 1.32.

A. LUMPED ELEMENT MODEL OF A CO-AXIAL TRANSMISSION LINE

A.1 The speed of wave propagation in free space ($c_0 = 299\,792\,458\,\mathrm{m/s}$) is $c_0 = 1/\sqrt{\varepsilon_0\,\mu_0}$. The speed in the dielectric & diamagnetic medium is

$$\boxed{v = \frac{c_0}{\sqrt{\varepsilon_\mathrm{r}\,\mu_\mathrm{r}}}} \tag{A.1}$$

A.2 Gauss law for the flux through a cylindrical surface with radius r co-axial with the the core, $a < r < b$:

$$\Delta x\, 2\pi r\, E(r) = \frac{\Delta q}{\varepsilon_\mathrm{r}\varepsilon_0} \Rightarrow E(r) = \frac{\Delta q}{\Delta x}\frac{1}{2\pi\varepsilon_\mathrm{r}\varepsilon_0 r} \tag{A.2}$$

A.3 The capacitance

$$C_x\,\Delta x = \frac{\Delta q}{\varphi} \tag{A.3}$$

where the potential φ of the core with respect to the shield is

$$0 - \varphi = -\int_a^b E(r)\,dr \Rightarrow \varphi = \frac{\Delta q}{\Delta x}\frac{1}{2\pi\varepsilon_\mathrm{r}\varepsilon_0}\ln\frac{b}{a} \tag{A.4}$$

$$\boxed{C_x = \frac{2\pi\varepsilon_\mathrm{r}\varepsilon_0}{\ln\frac{b}{a}}} \tag{A.5}$$

A.4 The magnetic flux through a rectangular contour paralel to the axis equal inductance times the current:

$$\Delta x \int_a^b B(r)\,dr = L_x\,\Delta x\, I \tag{A.6}$$

Biot-Savart law $B(r) = \frac{\mu_\mathrm{r}\mu_0}{2\pi}\frac{I}{r}$ gives

$$\boxed{L_x = \frac{\mu_\mathrm{r}\mu_0}{2\pi}\ln\frac{b}{a}} \tag{A.7}$$

A.5 i. Adding δx length of the cable should not change its impedance. Hence the impedance Z of the following circuit must be equal to Z_0:

$$\frac{1}{Z} = \frac{1}{Z_0 + j\omega\delta L} + \frac{1}{\frac{1}{j\omega\delta C}} = \frac{1}{Z_0} \tag{A.8}$$

$$Z_0^2 + j\,\omega\,\delta L\, Z_0 - \delta L/\delta C = 0 \tag{A.9}$$

(here engineering notation for $j^2 = -1$ is used.) $\delta L/\delta C = L_x/C_x$ and $\delta L \to 0$ for $\delta x \to 0$, hence

$$\boxed{Z_0 = \sqrt{L_x/C_x}} \tag{A.10}$$

ii.

$$Z_0 = \sqrt{L_x/C_x} = \frac{\ln(b/a)}{2\pi}\sqrt{\frac{\mu_\mathrm{r}\mu_0}{\varepsilon_\mathrm{r}\varepsilon_0}} = \ln(b/a)\sqrt{\frac{\mu_\mathrm{r}}{\varepsilon_\mathrm{r}}} \times 59.96\,\Omega \tag{A.11}$$

For $Z_0 = 50\,\Omega$, $\varepsilon_\mathrm{r} = 4.0$ and $\mu_\mathrm{r} = 1.0$ this gives $\boxed{b = 5.30\,a}$.

B. HYPOTHETICAL TRANSMISSION LINE WITH RETURN ALONG A GROUNDED PLANE

B.1 The high-conductance ground plate can be replaced by an image of the wire with opposite direction of the current at distance $2d$ from the real wire. The magnetic fields from the real and the imaginary wires add up and need to be integrated to get the magnetic flux between the wire and the plate:

$$L_x \, \Delta x \, I = \frac{\mu\mu_0}{2\pi} I \int_a^d \left(\frac{1}{r} + \frac{1}{2d-r} \right) dr \, \Delta x \tag{B.1}$$

$$L_x = \frac{\mu\mu_0}{2\pi} \ln \left(\frac{2d}{a} - 1 \right) \approx \frac{\mu\mu_0}{2\pi} \ln \frac{2d}{a} \tag{B.2}$$

The potential difference between the wire and the plate can be obtained similarly by integrating the combined field for the wire and its image:

$$\varphi = \frac{\Delta q}{\Delta x} \frac{1}{2\pi\varepsilon_\mathrm{r}\varepsilon_0} \int_a^d \left(\frac{1}{r} + \frac{1}{2d-r} \right) dr = \frac{\Delta q}{\Delta x} \frac{\ln(2d/a)}{2\pi\varepsilon_\mathrm{r}\varepsilon_0} \tag{B.3}$$

$$C_x = \frac{\Delta q}{\Delta x} \frac{1}{\varphi} \approx \frac{2\pi\varepsilon_\mathrm{r}\varepsilon_0}{\ln(2d/a)} \tag{B.4}$$

Hence the characterstic impedance $Z_0 = \sqrt{L_x/C_x}$ of the wire-plate system is

$$\boxed{Z_0 = \frac{\ln(2d/a)}{2\pi} \sqrt{\frac{\mu_\mathrm{r}\mu_0}{\varepsilon_\mathrm{r}\varepsilon_0}}} \tag{B.5}$$

C. BASICS OF RF REFLECTOMETRY

C.1 At the interface, values of the voltage on both transmission lines have to coincide:

$$V_\mathrm{i} + V_\mathrm{r} = V_\mathrm{t} \tag{C.1}$$

The current has to be conserved at the interface, however, the incident and the reflected waves carry the current in opposite directions:

$$\frac{V_\mathrm{i}}{Z_0} - \frac{V_\mathrm{r}}{Z_0} = \frac{V_\mathrm{t}}{Z_1} \tag{C.2}$$

It is clear from the equation above that $V_\mathrm{t} \neq 0$ if $Z_0 \neq Z_1$ – impedance mismatch has to cause reflection. Solving the voltage and the current equations for $\Gamma = V_\mathrm{r}/V_\mathrm{i}$ gives

$$\boxed{\Gamma = \frac{Z_1 - Z_0}{Z_1 + Z_0}} \tag{C.3}$$

C.2 A π-shift implies opposite signs of V_i and V_r and hence requires $\Gamma < 0$. This implies $\boxed{Z_1 < Z_0}$.

D. THE SINGLE ELECTRON TRANSISTOR

D.1 i. Since any capacitance beyond C_g is neglected in our model, the quantum dot can be thought as a capacitor plate with the gate being the other plate of the same capacitor with capacitance C_g. The fixed number n of electrons trapped on the quantum dot sets a fixed-charge ($q = -ne$) boundary condition for the capacitor C_g on the QD, while the gate side is kept at a constant potential V_g. (We denote the elementary charge by $e > 0$). The implies that an excess charge of opposite sign, $-q = ne$ will accumulate on the gate, to keep electric field confined between the QD and the gate. The potential jump across the capacitor from the gate to the QD will be equal to the capacitor $q/C_g = -ne/C_g$. Hence the potential on the QD is

$$\boxed{\varphi_n = V_g + \frac{-ne}{C_\mathrm{g}}} \tag{D.1}$$

ii. Bringing an infinitesimal charge δq from potential 0 to potential $\varphi(q)$ requires energy $\delta E = \varphi(q)\delta q$, and the dependence of potential $\varphi(q)$ on the accumulated charge q is linear. For the single-electron transfer, the additional charge of the electron, $-e$, changes the potential from φ_n to $\varphi_{n+1} = \varphi_n - e/C_g$. Hence the work necessary to accumulate an extra e on the QD is the integral of δE

$$\Delta E_n = -e\frac{\varphi_n + \varphi_{n+1}}{2} \tag{D.2}$$

$$\boxed{\Delta E_n = \frac{e^2}{C_g}\left(n + \frac{1}{2}\right) - eV_g} \tag{D.3}$$

Alternatively, ΔE_n can be obtained from energy conservation, by computing the change of the energy of the capacitor the dork the work done against the electromotive force of the battery (=− "work done by the battery') for a charge $+e$ to be brought from the ground potential via the battery to the gate-side plate of the capacitor:

$$\Delta E_n = \frac{e^2(n+1)^2}{2C_g} - \frac{e^2 n^2}{2C_g} - eV_g \tag{D.4}$$

Note that without $C_t \ll C_g$ approximation, the answer is $\Delta E_n = \frac{e^2}{C_g + 2\,C_t}\left(n + \frac{1}{2}\right) - eV_g C_g/(2\,C_t + C_g)$ (not required to receive full marks).

D.2 $\mathcal{N}$ is a minimal integer n for which $\Delta E_n \geq 0$. Consider the marginal case of $\Delta E_{\mathcal{N}} = 0$ which is achieved at some $V_g = V_0$,

$$\Delta E_{\mathcal{N}}(V_0) = 0 = \frac{e^2}{C_g}\left(\mathcal{N} + \frac{1}{2}\right) - eV_0 \tag{D.5}$$

If V_g would go slightly larger than V_0, then ΔE_n would go negative and then minimal n that makes a positive ΔE_n would jump from $\mathcal{N}$ to $\mathcal{N}+1$. Hence $E_c = \Delta E_{\mathcal{N}+1}(V_0)$. This gives

$$\Delta E_{\mathcal{N}+1}(V_0) = E_c = \frac{e^2}{C_g}\left(\mathcal{N} + 1 + \frac{1}{2}\right) - eV_0 = \boxed{\frac{e^2}{C_g}} \tag{D.6}$$

D.3 In a metal, only electrons in an energy range $\pm \approx k_B T$ around the Fermi level take part in the thermal motion. (Here k_B is the Boltzmann constant.) Typical energy of these electrons is $k_\mathrm{B}T$ per particle and it may not exceed characteristic single-electron addition energy E_c, $\boxed{k_\mathrm{B}T < E_c}$.

D.4 i. $\boxed{\tau = R_t\,C_t}$

ii. Quantum uncertainty of energy (life-time broadening) h/τ must be less than the energy difference between the states with n and $n+1$ electrons,

$$h/\tau < E_c \Rightarrow \frac{h}{R_t C_t} < \frac{e^2}{C_g} \tag{D.7}$$

$$\boxed{R_t > \frac{h}{e^2}\frac{C_g}{C_t}} > \frac{h}{e^2} \tag{D.8}$$

E. RF REFLECTOMETRY TO READ OUT SET STATE

E.1

$$\Gamma = \frac{Z_\mathrm{SET} - Z_0}{Z_\mathrm{SET} + Z_0} \tag{E.1}$$

$$\Gamma_\mathrm{ON} = \frac{10^5 - 50}{10^5 + 50} \approx 1 - 2\frac{50}{10^5} \tag{E.2}$$

$$\Gamma_\mathrm{OFF} = \lim_{Z_1 \to \infty} \frac{Z_1 - Z_0}{Z_1 + Z_0} = 1 \tag{E.3}$$

$$\Delta\Gamma = |\Gamma_\mathrm{ON} - \Gamma_\mathrm{OFF}|\boxed{\approx 1.0 \cdot 10^{-3}} \tag{E.4}$$

E.2 Large change in reflectance requires the impedance Z_1 of the circuit to switch between $Z_1 < Z_0$ to $Z_1 > Z_0$ as the SET between ON ($Z_{\mathrm{SET}} = 100\mathrm{k}\Omega$) and OFF ($Z_{\mathrm{SET}} = \infty$).

In the OFF state of the SET, the circuit is an disspationless LC contour with resonance frequency $\omega_0 = 1/\sqrt{L_0 C_0}$ and its impedance is 0. If we choose

$$\boxed{L_0 = \frac{1}{\omega_{\mathrm{rf}}^2 C_0}} \tag{E.5}$$

then the imedance of the $\omega_0 = \omega_{\mathrm{rf}}$.

Since Z_{tot} (the total impedance of the circuit) in the OFF state of the SET equals to 0, the reflectance i $\Gamma_{\mathrm{OFF}} = -1$. As we switch to the ON state with $Z_{\mathrm{SET}} = R_{\mathrm{SET}} = 10^5\Omega$, the change in reflectance will be large if $|Z_{\mathrm{tot}}|$ in this ON state is on the order of Z_0 or larger, which is indeed the case.

For the ON state and $\omega_0 = \omega_{\mathrm{rf}}$

$$Z_{\mathrm{tot}} = \left(\frac{1}{\frac{1}{j\,\omega\,C_0}} + \frac{1}{R_{\mathrm{SET}}}\right)^{-1} + j\,\omega L_0 = \frac{R_{\mathrm{SET}}}{1 + j\,\omega C_0\,R_{\mathrm{SET}}} + j\,\omega\,L_0 = \frac{R_{\mathrm{SET}} + j\,\sqrt{L_0/C_0}}{1 + R_{\mathrm{SET}}^2 C_0/L_0} \tag{E.6}$$

For $C_0 = 0.4 \cdot 10^{-12}\,\mathrm{F}$, $Z_0 = 50\,\Omega$ and $\omega_{\mathrm{rf}} = 2\pi \cdot 10^8\,\mathrm{Hz}$, we have $\boxed{L_0 = 6.33\,\mu\mathrm{H}}$, $Z_{\mathrm{tot}} = (158 + 6.3\,j)\Omega$, $\Gamma_{\mathrm{ON}} = 0.5198 + 0.0145\,j$, and $\boxed{\Delta\Gamma = 1.52}$.

F. CHARGE SENSING WITH A SINGLE LEAD QUANTUM DOT

F.1 The SLQD readout circuit contains only reactive elements, so $|\Gamma| = 1$ will always be one. The OFF state of the SLQD corresponds to an inductor L_0 and a capacitor C_0 connected in parallel. We again choose

$$\boxed{\omega_{\mathrm{rf}} = 1/\sqrt{L_0 C_0}} \tag{F.1}$$

so that Z_{tot} is the OFF state is infinite and $\Gamma_{\mathrm{OFF}} = 1$.

The ON state corresponds to $Z_{\mathrm{SET}} = -j\frac{1}{\omega_{\mathrm{rf}} C_q}$ and Z_{tot} at $\omega_{\mathrm{rf}} = \omega_0$ is just the impedance of the SLQD

$$Z_{\mathrm{tot}} = \frac{1}{(j\omega_{\mathrm{rf}} L_0)^{-1} + j\omega_{\mathrm{rf}}(C_0 + C_q)} = -j\frac{1}{\omega_0 C_q} = -j\frac{C_0}{C_q} Z_C \tag{F.2}$$

For the complex phase of $\Gamma_{\mathrm{ON}} = (Z_{\mathrm{tot}} - Z_0)/(Z_{\mathrm{tot}} + Z_0)$ to be significantly different from zero, we need $|Z_{\mathrm{tot}}| \sim Z_0$ since Z_{tot} is purely imaginary. Hence

$$\boxed{Z_C \sim \frac{C_q}{C_0} Z_0} \tag{F.3}$$

F.2 If L_0 is fixed, we can still operate the circuit at the frequency

$$\boxed{\omega_{\mathrm{rf}} = 1/\sqrt{L_0 C_0}} \tag{F.4}$$

that gives $\Gamma_{\mathrm{OFF}} = 1$. However, we need to deduce a way to increase $|Z_{\mathrm{tot}}|$ even if $Z_C \ll C_q Z_0/C_0$ is not sufficient. One of the ways to do that is to add an additional capacitance C_m is series with rest of the circuit. This will give (at $\omega_{\mathrm{rf}} = \omega_0$)

$$Z_{\mathrm{tot}} = -j\left(\frac{C_0}{C_q} Z_C + \frac{1}{\omega_0 C_m}\right) = -j\omega_0^{-1}\left(C_q^{-1} + C_m^{-1}\right) \tag{F.5}$$

We can satisfy the condition $|Z_{\mathrm{tot}}| = Z_0$ (and hence $\Gamma_{\mathrm{ON}} = j$ and $\Delta\Gamma = \sqrt{2} \sim 1$) with

$$C_m = \frac{C_q}{Z_0 C_q \omega_{\mathrm{rf}} - 1} = \frac{C_q\sqrt{L_0 C_0}}{Z_0 C_q - \sqrt{L_0 C_0}} \tag{F.6}$$

$$\boxed{C_m = \frac{C_q Z_C}{Z_0 C_q/C_0 - Z_C}} \overset{Z_C \ll Z_0 C_q/C_0}{\approx} \frac{1}{Z_0\,\omega_{\mathrm{rf}}} \tag{F.7}$$

X-ray jets from active galactic nuclei

Introduction

Active galactic nuclei (AGN) are supermassive black holes which form the centres of galaxies, and emit large amounts of energy in radiation and particle flows. One feature of many AGN are jetted outflows, which can be observed through radio emission, and sometimes also in other parts of the electromagnetic spectrum, including x-rays. These jets are large flows of plasma at relativistic speeds, over lengths of order 10^{20} m, which is tens of thousands of light years. The x-ray emission from jets is usually dominated by synchrotron emission from relativistic electrons gyrating in the magnetic field of the jet.

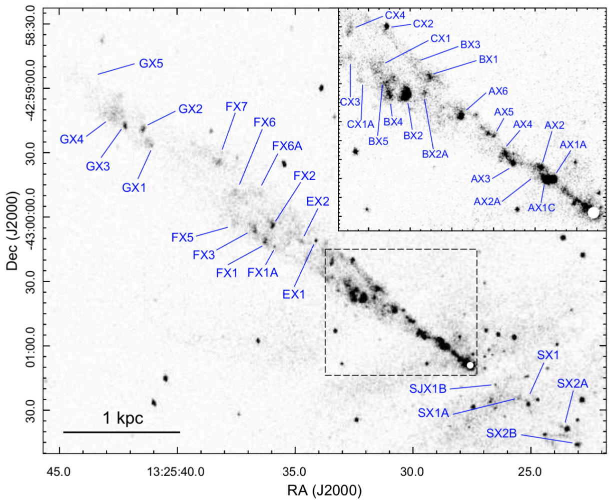


Figure 1: X-ray image of the jet from the Centaurus A AGN. Darker regions represent regions of higher intensity x-rays. Brighter regions within the fainter jet are called knots. (Snios *et al.*, 2019)

Part A: 1D fluid model of a jet

A simple model of the flow of jets assumes that the flow is steady and directed radially away from the central AGN, so approximately one dimensional, and that the plasma in the jet is in pressure equilibrium with its surroundings. There is assumed to be a constant rate per volume of mass injected into the jet from stars which lose their outer layers as they move through their life cycle.

The jet is described in terms of the coordinate representing distance from the AGN, s, and the opening radius r of the conical jet. These distances are measured in parsecs, where $1\ \mathrm{pc} = 3.086 \times 10^{16}\ \mathrm{m}$. The speed of the jet flow is assumed to be directed radially away from the central AGN, and be a function of s only. The plasma in the jet is comprised of electrons, protons, and some heavier ionised nuclei. The average energy carried by each particle in the jet, in the reference frame of the bulk flow of the jet (which we will call the jet frame), is $\epsilon_{\mathrm{av}} = \mu_{\mathrm{pp}} c^2 + h$, where the term h includes all thermal kinetic energy and potential energies in terms of the pressure P and n is the number density of the plasma.

As the stars, which the jet flows past, move through their life cycles they can lose part of their atmosphere. This results in a uniform rate of injection of mass per unit volume α into the jet, and the injected particles are assumed to be at rest relative to the AGN.

This model can be applied to the Centaurus A jet. Centaurus A is one of the nearest AGN, so it is possible to observe its jet at relatively high spatial resolution. The total power carried by the jet is estimated to be $P_{\mathrm{j}} = 1 \times 10^{36}$ J.s^{-1}. See below for a diagram of a simple geometrical description of the Centaurus A jet, including measurements of some jet parameters. s_1 is the coordinate of the start of the jet, and s_2 the coordinate of the end of the jet. In Centuarus A the average mass per particle is $\mu_{\mathrm{pp}} = 0.59 m_{\mathrm{p}}$ and $h = \frac{13}{4} P/n$ .The pressure in the plasma surrounding the jet is $P(s) = 5.7 \times 10^{-12} \left(\frac{s}{s_0}\right)^{-1.5}$ Pa, where $s_0 = 1$ kpc.

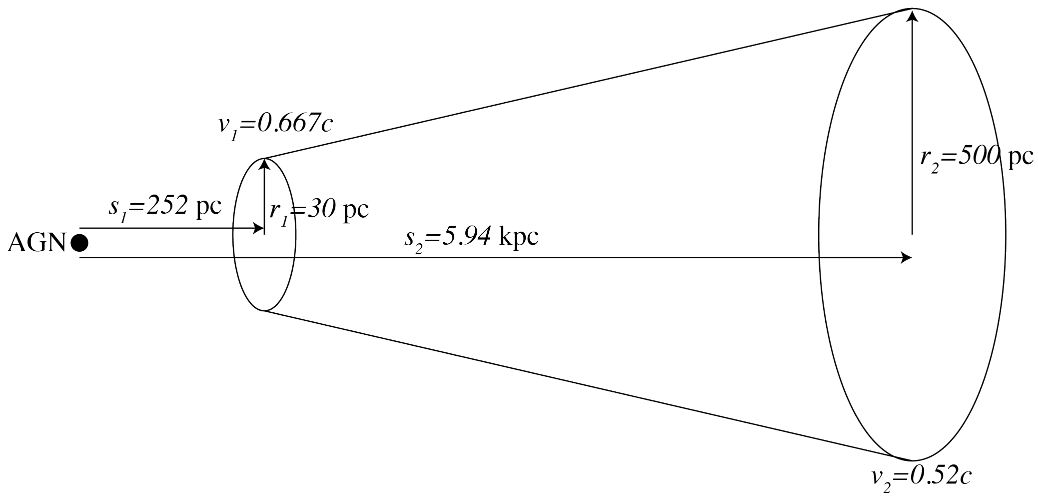


Figure 2: The Centaurus A jet, showing the geometry compared to the active galactic nucleus (AGN).

The jet is described by the following parameters, all of which depend on the distance s from the AGN:

- the opening radius of the jet $r(s)$ in the AGN frame
- the cross sectional area of the jet $A(s)$ in the AGN frame
- the speed of the jet $v(s)$ in the AGN frame
- the lorentz gamma factor of the jet $\gamma(s)$ in the AGN frame
- the number density $n(s)$ in the frame of the jet

Any of these parameters can be used in your answers to A1-4.

A.1	Find the number density of particles, $n'(s)$, in the frame of the AGN, in terms of the proper number density, $n(s)$ and other jet parameters. The proper number density is the number density in the frame which is locally co-moving with the jet plasma outflow, which we will call the jet frame.	0.3pt

A.2	Find the flux of particles, $F_p(s)$, across a cross section of the jet with area A, at a distance s from the AGN.	0.2pt

A.3	Write a continuity relationship between the particle flux into the jet and out of the jet in terms of the jet parameters at s_1 and s_2, and V, the total volume of the Centaurus A jet and other required parameters.	0.5pt

A.4	Write a relationship between the energy flux into the jet, and the energy flux out of the jet in terms of the jet speeds, cross sectional areas and proper number densities at $s_1$and s_2, the volume, V, of the jet and any other required parameters of the Centaurus A jet.	0.6pt

The power carried by a jet is defined to be the sum of the total bulk kinetic energy flux and the total thermal energy flux, so

$$P_{\mathrm{j}}(s) = F_{\mathrm{E}}(s) - \dot{M}c^2 \tag{1}$$

where $F_{\mathrm{E}}(s)$ is the flux of energy through the cross section of the jet at s, and $\dot{M}$ is the mass flux through the jet cross section at the same distance s from the AGN.

A.5	Using your answers to previous parts find $\frac{dP_{\mathrm{j}}}{ds}$.	0.6pt

A.6	Find numerical values for the mass fluxes $\dot{M}_1$, into the Centaurus A jet at s_1, and also $\dot{M}_2$, out of the Centaurus A jet at s_2,	0.4pt

A.7	Find an expression for the total momentum flux, Π, into the Centaurus A jet. Also numerically evaluate this expression.	0.5pt

A.8	Find a numerical value for the total force due to external pressure, F_{Pr}, on the Centaurus A jet.	0.5pt

A.9	Write the expected relationship between Π and F_{Pr}. Also, calculate the percentage difference between the model value of Π , which you found in A7, and the expected value.	0.2pt

Part B: Gas of ultra relativistic electrons

Consider a gas of ultra relativistic electrons ($\gamma \gg 1$), with an isotropic distribution of velocities (does not depend on direction). The proper number density of particles with energies between ϵ and $\epsilon + d\epsilon$ is given by $f(\epsilon)d\epsilon$, where ϵ is the energy per particle. Consider also a wall of area ΔA, which is in contact with the gas.

B.1	Write an integral expression for the total energy per volume of the electron gas.	0.2pt

B.2	Find an expression for the total rate of change in momentum $\Delta p_z/\Delta t$ of the gas, in the z-direction which is normal to the wall, due to collisions with the wall.	0.8pt

B.3	Derive an equation of state for an ultra relativistic electron gas, relating the pressure, volume and total internal energy.	0.6pt

B.4	Derive a relationship between the pressure and volume of an ultra relativistic electron gas undergoing an adiabatic expansion.	0.6pt

Part C: Synchrotron emission

In the jets from AGN, we have populations of highly energetic electrons in regions with strong magnetic fields. This creates the conditions for the emission of high fluxes of synchrotron radiation. The electrons are often so highly energetic, that they can be described as ultra relativistic with $\gamma \gg 1$.

C.1	Find an expression for Ω, the angular frequency of gyration of an electron with lorentz factor γ and travelling at an angle ϕ to the magnetic field B.	0.7pt

As the electron is accelerated due to the magnetic field it emits electromagnetic radiation. In a frame at which the electron is momentarily at rest, there is no preferred direction for the emission of the radiation. Half is emitted in the forward direction, and half in the backward direction. However, in the frame of the observer, for an electron moving at an ultra relativistic speed, with $\gamma \gg 1$, the radiation is concentrated in a forward cone with $\theta \lesssim 1/\gamma$ (so the total angle of cone is $2/\gamma$). As the electron is gyrating around the magnetic field, any observer will only see pulses of radiation as the forward cone sweeps through the line of sight.

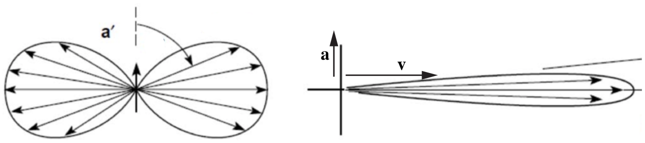


Figure 3: The diagram on the left shows the distribution of power in radiation from an electron accelerating up the page in the frame at which the electron in momentarily at rest. The diagram on the right shows the distribution of power in radiation for the same electron in the observer's frame, where most radiation is emitted in the forward cone. In the observers frame, the direction of the electron's acceleration is shown by a vector labelled **a** and the direction of its velocity is shown by a vector labelled **v**.

C.2	Find the duration of a pulse, Δt, of synchrotron radiation observed from an electron with lorentz factor γ, travelling at an angle ϕ to the magnetic field.	0.5pt

C.3	Hence, estimate the characteristic frequency, ν_{chr}, of the synchrotron radiation.	0.3pt

The total synchrotron power emitted is

$$P_{\text{s}} = \frac{1}{6\pi\varepsilon_0}\left(\frac{q^4 B^2 \sin^2\phi}{m^4 c^5}\right) E^2 \tag{2}$$

C.4	Estimate the time, τ, for an electron of energy E to lose its energy through synchrotron cooling.	0.2pt

Part D: Synchrotron emission from an AGN jet

The distribution of electron energies in a jet from an AGN is typically a power law, of the form $f(\epsilon) = \kappa\epsilon^{-p}$, where $f(\epsilon)d\epsilon$ is the number density of particles with energies between ϵ and $\epsilon + d\epsilon$. The corresponding spectrum of synchrotron emission depends on the electron energy distribution, rather than the spectrum for an individual electron. This spectrum is

$$j(\nu)d\nu \propto B^{(1+p)/2}\nu^{(1-p)/2}d\nu \quad . \tag{3}$$

Here $j(\nu)d\nu$ is the energy per unit volume emitted as photons with frequencies between ν and $\nu + d\nu$

Observations of the Centaurus A jet, and other jets, show a knotty structure, with compact regions of brighter emission called knots. Observations of these knots at different times have shown both motion and brightness changes for some knots. Two possible mechanisms for the reductions in brightness are adiabatic expansion of the gas in the knot, and synchrotron cooling of electrons in the gas in knot.

The magnetic field in the plasma in the jets is assumed to be *frozen in.* Considering an arbitrary volume of plasma, the magnetic flux through the surface bounding it must remain constant, even as the volume containing the plasma changes shape and size.

D.1	For a spherical knot which expands uniformly in all directions from a volume of V_0 to a volume V, with an initial uniform magnetic field B_0 Find the magnetic field B in the expanded knot.	0.4pt

D.2	Find $f(\epsilon)$, the distribution of electron energies after adiabatic expansion of a spherical knot to a volume V on the distribution of electron energy densities, given that the knot of volume V_0 has an initial distribution of electrons $f_0(\epsilon) = \kappa_0\epsilon^{-p}$, where $f_0(\epsilon)d\epsilon$is the number density of particles with energies between $\epsilon$and $\epsilon + d\epsilon$.	1.0pt

D.3	How will synchrotron cooling affect the distribution of the electrons? After a time interval where electrons have been undergoing synchrotron cooling, will the distribution of electron energies as a function of ϵ be steeper, shallower or leave it unchanged. Justify your answer with equations, by considering two electron energies $\epsilon_1 < \epsilon_2$.	0.3pt

The table below summarises some observations of knots (brighter regions) in jets from two AGN, Centaurus A (Cen A) and M87.

AGN	Time between observations	Knot	Brightness change in x-rays	Spectral changes in x-rays	Brightness changes in other bands (e.g. UV, optical)
Cen A	15 years	AX1C	-23%	No change	No data
Cen A	15 years	BX2	-15%	No change	No data
M87	5 years	HST-1	-73%	No data	No change
M87	5 years	Knot A	-12%	No data	No change

(Data from Snios et *al.*, 2019a; 2019b.)

D.4	In the table in the answer sheet, identify the more likely cause of reduced brightness for each knot, and identify which previous part or parts support your conclusion.	0.6pt

Theory
Question 2: X-ray jets from active galactic nuclei
Solutions

Part A: 1d fluid model of a jet

A1

If you consider a prism of plasma in the jet frame, it contains a number of particles N, has length l in the direction of motion, and cross sectional area A. The total number of particles in the volume is invariant on transformation into the AGN frame, however the volume occupied by the plasma changes as lengths are contracted in the direction of motion, while perpendicular lengths are unchanged. Hence, $A' = A$, and $l' = l/\gamma$.

This gives us two relationships:

$$N = n(s)Al \tag{1}$$

and

$$N = n'(s)Al/\gamma \tag{2}$$

Equating these gives

$$n(s)Al = n'(s)Al/\gamma \quad ,$$

which leads to

$$n'(s) = \gamma n(s) \quad . \tag{3}$$

A2

The particles in the jet have a bulk flow speed of $v(s)$, so in a time Δt a volume $V = A(s)v(s)\Delta t$ crosses the cross section of the jet. Using the number density in the AGN frame,

$$F_{\mathrm{p}}(s) = n'(s)A(s)v(s) \tag{4}$$
$$= \gamma(s)n(s)A(s)v(s) \tag{5}$$

A3

As the plasma travels along the jet there are no particles passing through the side boundary of the jet. Hence, the total flux through the curved edges of the jet is zero, and the total flux into the jet is the flux in through the cross section at s_1 is $F_{\mathrm{p}}(s_1)$ and the total flux out of the jet is $F_{\mathrm{p}}(s_2)$. There is an additional term in the continuity equation due to the mass injection. There are $\alpha V/\mu_{\mathrm{pp}}$ particles injected.

This gives

$$\gamma(s_2)v(s_2)n(s_2)A(s_2) - \gamma(s_1)v(s_1)n(s_1)A(s_1) = \alpha V/\mu_{\mathrm{pp}} \tag{6}$$

A4

Similarly, in the AGN frame the energy flux

$$F_{\mathrm{E}}(s) = n'(s)A'(s)v(s)\epsilon'_{\mathrm{av}}(s) \quad . \tag{7}$$

We use previous results for all quantities except average energy per particle.

Consider the total energy in a volume ΔV of the plasma, $E_{\mathrm{tot}} = \epsilon_{\mathrm{av}}N$ in the jet frame. As this is the proper frame v(s)=0.

Transforming to the AGN frame, $E'_{\mathrm{tot}} = \gamma(s)\epsilon_{\mathrm{av}}N$, and $\epsilon'_{\mathrm{av}} = \gamma\epsilon_{\mathrm{av}}$.

Hence,

$$F_{\mathrm{E}}(s) = (\gamma(s))^2 n(s)A's)v(s)\epsilon_{\mathrm{av}}(s) \quad . \tag{8}$$

Energy conservation requires that the total energy flux out of the jet is equal to the energy added through injection of mass, so

$$(\gamma(s_2))^2 v(s_2)n(s_2)A(s_2)\epsilon_{\mathrm{av}}(s_2) - (\gamma(s_1))^2 v(s_1)n(s_1)A(s_1)\epsilon_{\mathrm{av}}(s_1) = \alpha V c^2 \tag{9}$$

A5

From the defintion of jet power and also (8),

$$P_j(s) = (\gamma(s))^2 \, n(s)A's)v(s)\epsilon_{\rm av}(s) - \dot{M}c^2 \quad . \tag{10}$$

Here $\dot{M}$ is the flux of mass flux across the surface, so $\dot{M} = F_{\rm p}(s)\mu_{\rm pp}$ and

$$P_j(s) = (\gamma(s))^2 \, n(s)A's)v(s)\epsilon_{\rm av}(s) - F_{\rm p}(s)\mu_{\rm pp}c^2 \quad . \tag{11}$$

In order to find how jet power varies along the jet, we consider jet power at two points along the jet.

$$P_{\rm j}(s_2) - P_{\rm j}(s_1) = (\gamma(s_2))^2 \, n(s_2)A'(s_2)v(s_2)\epsilon_{\rm av}(s_2) - F_{\rm p}(s_2)\mu_{\rm pp}c^2 \tag{12}$$

$$- \left((\gamma(s_2))^2 \, n(s_1)A'(s_1)v(s_1)\epsilon_{\rm av}(s_1) - F_{\rm p}(s_1)\mu_{\rm pp}c^2\right) \quad . \tag{13}$$

We can identify the two terms with $\epsilon_{\rm av}$ to be those from the left hand side of (8), and the two terms with $\mu_{\rm pp}$ are $\mu_{\rm pp}c^2$ times the left hand side of (6). Making these substitutions,

$$P_{\rm j}(s_2) - P_{\rm j}(s_1) = \alpha V c^2 - \alpha V c^2 = 0 \quad . \tag{14}$$

This argument applies to arbitary s_1 and s_2, so the jet power is constant along the jet and $\frac{dP_{\rm j}}{ds} = 0$.

A6

We start from (10) and substitute $\epsilon_{\rm av} = \mu_{\rm pp}c^2 + \frac{13}{4}\frac{P}{n}$, to arrive at

$$P_{\rm j}(s) = (\gamma(s))^2 \, n(s)A(s)v(s)(\mu_{\rm pp}c^2 + \frac{13}{4}\frac{P}{n(s)}) - \gamma(s)n(s)A(s)v(s)\mu_{\rm pp}c^2 \tag{15}$$

$$= (\gamma(s) - 1)\gamma(s)n(s)A(s)v(s)\mu_{\rm pp}c^2 + (\gamma(s))^2 \, A(s)v(s)\frac{13}{4}P \tag{16}$$

$$= (\gamma(s) - 1)\dot{M}c^2 + (\gamma(s))^2 \, A(s)v(s)\frac{13}{4}P \tag{17}$$

Rearranging to find $\dot{M}$ gives

$$\dot{M} = \frac{P_{\rm j} - \gamma(s)^2 A(s)v(s)\frac{13}{4}P}{(\gamma(s) - 1)c^2} \tag{18}$$

Using the relationship $P(s) = 5.7 \times 10^{-12} \left(\frac{s}{s_0}\right)^{-1.5}$ and substituting values for s_1 and s_2 respectively into (18), give $\dot{M}_1 = 2.8 \times 10^{19}$ kg s^{-1} and $\dot{M}_2 = 5.2 \times 10^{19}$ kg s^{-1} .

Note: some of the input values are given to one significant figure only. Hence, answers which are correct to this degree of precision and are given to one or two significant figures are accepted as correct.

A7

From lorentz transforming $\epsilon_{\rm av}$ from the jet frame where $v = 0$ to the AGN frame, the average momentum per particle is $p_{\rm av} = \gamma(s)\frac{v(s)}{c^2}\epsilon_{\rm av}$. As the momentum is directly proportional to the total energy, the flux argument is the same, and

$$\Pi(s) = \frac{F_{\rm E}}{c}\frac{v(s)}{c} \quad . \tag{19}$$

This can be related to the jet power and $\dot{M}$,

$$\Pi(s) = \left(\frac{P_{\rm j}}{c} + \dot{M}c\right)\frac{v(s)}{c} \quad . \tag{20}$$

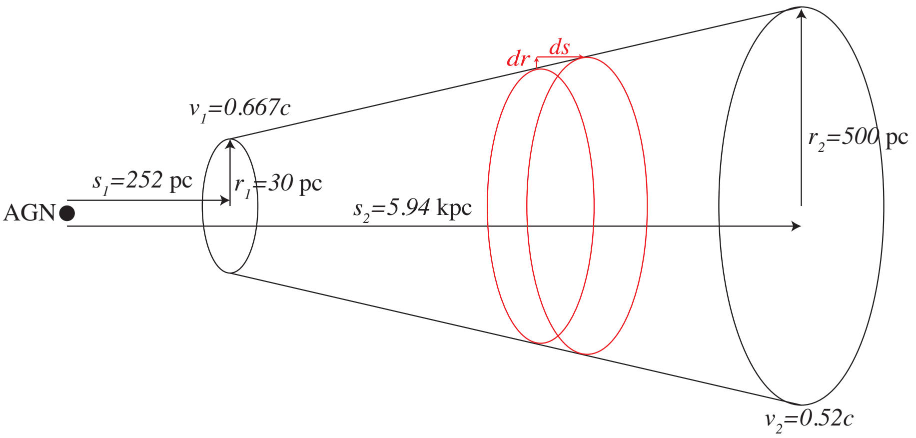
Again, there is no particle flux, and hence no momentum flux through the sides of the jet, so the total momentum flux out of the jet is

$$\Pi = \Pi(s_2) - \Pi(s_1) \quad . \tag{21}$$

Substituting values for the jet at s_2 and s_1 gives $\Pi = 1.9 \times 10^{27}$ kg m s^{-2}.

A8

The total force on the jet due to external pressure has contributions from the cross section at s_1, $F_1 = P(s_1)A(s_1)$, at s_s, $F_2 = P(s_2)A(s_2)$, and from the pressure on the curved surface. We have a linear relationship $s(r) = s_1 + \frac{s_2-s_1}{r_2-r_1}(r-r_1)$.



The nett pressure force on the surface is only the component in the s direction. As the force is perpendicular to the surface, this results in a factor of $\frac{dr}{ds}$. Consequently

$$dF = 2\pi r P(s) dr \quad , \tag{22}$$

where $P(s) = 5.7 \times 10^{-12} \left(\frac{s}{s_0}\right)^{-1.5}$.

The total force due to the external pressure,

$$F_{\mathrm{Pr}} = F_1 - F_2 + \int_{r_1}^{r_2} dF \quad . \tag{23}$$

Evaluating the integral gives $\int_{r_1}^{r_2} dF = 9.8 \times 10^{26}$ N, so $F_{\mathrm{Pr}} = 8.2 \times 10^{26}$ N.

A9

As there are no other forces on the jet, it is expected that $\Pi = F_{\mathrm{Pr}}$.

The % deviation is $|(\Pi - F_{\mathrm{Pr}})/F_{\mathrm{Pr}}| \approx 40\%$

Gas of ultrarelativistic electrons

B1

The total energy per volume is

$$\int_0^\infty \epsilon f(\epsilon) d\epsilon$$

B2

Consider the particles colliding with a surface ΔA,with the normal to the surface in the z-direction, in time Δt. As the electrons are ultrarelativistic, theirs speeds are all approximately c. We assume that the collisions with wall are elastic, and electrons depart with their parallel mometnum unchanged and $p_{z,\,\mathrm{final}} = -p_z$. Hence, $\Delta p_z = 2p_z$, where $p_z = \frac{\epsilon}{c}\cos\theta$, since the electrons are ultrarelativistic and $E \approx pc$.

The distribution is isotropic so electrons are equally likely to be travelling in any direction.

All electrons within a parallelepiped of length $c\Delta t$ which approach the surface at an angle θ will hit it in the time Δt. The volume of the paralleleiped is $c\Delta t \Delta A \cos\theta$. From here, the total change in momentum is

$$\Delta p_z = \int_0^\infty \int_0^{\pi/2} \int_0^{2\pi} 2f(\epsilon) p_z c\Delta t \Delta A \cos\theta \frac{\sin\theta}{4\pi} d\phi d\theta d\epsilon \tag{24}$$

$$= \frac{2\Delta t \Delta A}{4\pi} \int_0^{\pi/2} \sin\theta \cos^2\theta d\theta \int_0^{2\pi} d\phi \int_0^\infty \epsilon f(\epsilon) d\epsilon \tag{25}$$

$$= \frac{2\Delta t \Delta A}{4\pi} \times \frac{1}{3} \times 2\pi \int_0^\infty \epsilon f(\epsilon) d\epsilon \tag{26}$$

B3

As the remaining integral in the expression above was identified as the energy per volume in B1, $\Delta p_z = \Delta t \Delta A \frac{1}{3}\frac{E}{V}$. The pressure is the force per area normal to the wall, so $P = \frac{\Delta p_z}{\Delta t}\frac{1}{\Delta A}$. Combining these gives $P = \frac{E}{3V}$, or $E = 3PV$, which is the equation of state.

B4

For an adiabatic process $dQ = 0$ so $dE = dW = -PdV$. $dE = d(3PV) = 3PdV + 3VdP$, so equating these expressions gives

$$3PdV + 3VdP = -pdV \tag{27}$$

$$4PdV = -3VdP \tag{28}$$

$$4\frac{dV}{V} = -3\frac{dP}{P} \tag{29}$$

$$4\int_{V_0}^{V}\frac{dV'}{V'} = -3\int_{P_0}^{P}\frac{dP'}{P} \tag{30}$$

$$4\ln\left(\frac{V}{V_0}\right) = -3\ln\left(\frac{P}{P_0}\right) \tag{31}$$

$$\frac{PV^{4/3}}{P_0V_0^{4/3}} = 1 \tag{32}$$

Synchrotron emission

C1

An electron in a magnetic field has a component of its velocity, $v\cos\phi$ along the magnetic field, and $v\sin\phi$ perpendicular to the field. The parallel component of the velocity remains constant, but in the perpendicular direction the electron experiences a force in a direction perpendicular to its motion, so it undergoes simple harmonic motion. The perpendicular component of its velocity is Ωr where Ω is its angular frequency and r the radius of the circular motion. The force on the electron is $\mathbf{F}_\mathrm{B} = q\mathbf{v}\times\mathbf{B} = e\Omega rB\sin\phi$. The acceleration of the electron is perpendicular to the direction of motion, so $F_B = \gamma ma$, where a is the acceleration and m the mass of the electron. For uniform circular motion, $a = -\Omega^2 r$, so

$$F_\mathrm{B} = \gamma m\Omega^2 r \tag{33}$$

$$e\Omega rB\sin\phi = \gamma m\Omega^2 r \tag{34}$$

$$\Omega = \frac{eB\sin\phi}{\gamma m} \tag{35}$$

C2

The observer only sees the synchrotron emission when they are within the forward light cone. As the electron is gyrating around the magnetic field, this direction is changing. The observer is in this light cone for time $\Delta t = \frac{2\theta}{\Omega} = \frac{2m}{eB}$. However, the emitting electron is moving directly toward the observer over this time, so although the light emitted at the start of the pulse is ahead of the light at the end of the pulse, it is only ahead by $c\Delta t\left(1-\frac{v}{c}\right)$. The pulse then has an apparent duration of

$$\Delta t_a = \Delta t\left(1-\frac{v}{c}\right) \quad .$$

Since $\left(1-\frac{v}{c}\right)\left(1+\frac{v}{c}\right) = 1-\frac{v^2}{c^2} = \frac{1}{\gamma^2}$, we can write $\left(1-\frac{v}{c}\right) = \frac{1}{\gamma^2\left(1+\frac{v}{c}\right)}$. As the electrons are ultrarelativistic, $\left(1+\frac{v}{c}\right) = 2$, and

$$\Delta t_\mathrm{a} = \frac{m_e}{\gamma^2 eB} \quad .$$

C3

$$\nu_\mathrm{chr} \approx \frac{1}{\Delta t_\mathrm{a}} = \frac{\gamma^2 eB}{m_e}$$

C4

Making a linear approximation,

$$\tau \approx -\frac{E}{\left(\frac{dE}{dt}\right)} \tag{36}$$

$$= \frac{6\pi\varepsilon_0 m^4 c^5}{e^4 B^2 \sin^2\phi}\frac{1}{E} \tag{37}$$

Synchrotron emission from an AGN jet

D1

As the magnetic field is frozen in, and magnetic flux is constant, the magnetic field must decrease as the area increases in the expansion.

For a small area A, $B_0 A_0 = BA$. Since $A \propto V^{2/3}$, $B = B_0(A_0/A) = B_0\left(\frac{V}{V_0}\right)^{-2/3}$

D2

A volume of plasma V_0 with number density n_0 contains a total number of particles $N = n_0 V_0$. As the volume expands, the total number remains constant, so $n = N/V = (V/V_0)n_0$.

The internal energy of the plasma $E = 3PV$, and since $PV^{4/3} = P_0 V_0^{4/3}$, $EV^{1/3} = E_0 V_0^{1/3}$. The scaling for particle energy with volume is then $E = (V/V_0)^{-1/3}E_0$. This means that the particles initially with energies between ϵ_0 and $\epsilon + d\epsilon$, will have energies between $(V/V_0)^{-1/3}\epsilon_0$ and $(V/V_0)^{-1/3}(\epsilon + d\epsilon)$. As $((V/V_0)^{-1/3}\epsilon)^{-p} = (V/V_0)^{-p/3}\epsilon^{-p}$.

Hence, we can write

$$f(\epsilon) = \kappa\epsilon^{-p} \quad .$$

The value of κ is determined by the relationship

$$\int_0^\infty \kappa\epsilon^{-p} d\epsilon = N/V \quad .$$

Given

$$\int_0^\infty \kappa_0\epsilon^{-p} d\epsilon = N/V_0$$

$\kappa_0 V_0 = \kappa V$, and

$$f(\epsilon) = \left(\frac{V}{V_0}\right)^{-1} \kappa_0\epsilon^{-p}$$

D3

As the energy loss rate due to synchrotron emission increases as E^2, and the cooling time decreases as $1/E$, the more energetic electrons lose energy more rapidly. If we consider electrons with energies $\epsilon_1 < \epsilon_2$, both will move to lower energies in the distribution, but $df/dt \propto E^2$, so $\frac{df}{dt}|_{\epsilon_2} > \frac{df}{dt}|_{\epsilon_1}$. This will reduce the relative number of electrons with higher energies, and steepen the power law of the electron energy distribution.

D4

For the knots in Centaurus A there is no change in the x-ray spectrum, so this rules out synchrotron cooling as in that case the spectrum would steepen (Part D3). Hence adiabatic cooling is more likely for these two knots.

For the knots in M87, there is no change in brightness in other bands. Adiabatic expansion would reduce the number density at all energies (Part D2) and hence brightness at all wavelengths, so this is not likely. Hence, synchrotron cooling is more likely for these two knots.

Tippe top

Part A (10.0 points)

A Tippe top is a special kind of top that can spontaneously invert once it has been set spinning. One can model a Tippe top as a sphere of radius R that is truncated, with a stem added. It has rotational symmetry about an axis through the stem, which is at angle θ from the vertical. As shown in Figure 1(a), its centre of mass C is offset from its geometric centre O by αR along its symmetry axis. The Tippe top makes contact with the surface it rests on at point A; we assume this surface is planar, and refer to it as the floor. Given certain geometrical constraints and if spun fast enough initially, the Tippe top will tip so that the stem points increasingly downwards, until it starts to spin on in its stem, and eventually comes to a stop.

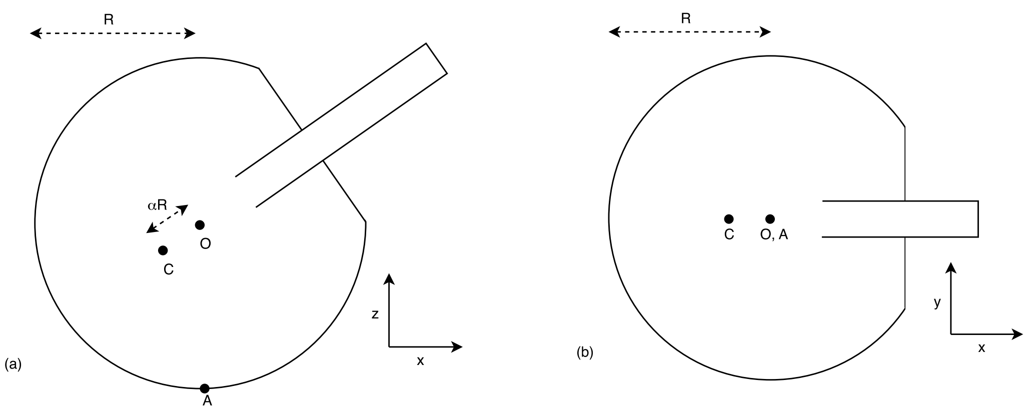


Figure 1. Views of the Tippe top (a) from the side and (b) from above

Let xyz be the rotating reference frame defined such that $\hat{\mathbf{z}}$ is stationary and upwards, and the top's symmetry axis is within the xz-plane. Two views of the Tippe top are shown in Figure 1: from the side, and from above. As shown in Figure 1(b), the top's symmetry axis is aligned with the x-axis when viewed from above.

Figure 2 shows the top's motion at several phases after it is started spinning:

(a) **phase I:** immediately after it is initially set spinning, with $\theta \sim 0$

(b) **phase II:** soon after, having tipped to angle $0 < \theta < \frac{\pi}{2}$

(c) **phase III:** when the stem first touches the floor, with $\theta > \frac{\pi}{2}$

(d) **phase IV:** after inversion, when the top is spinning on its stem, with $\theta \sim \pi$

(e) **phase V:** in its final state, at rest on its stem $\theta = \pi$.

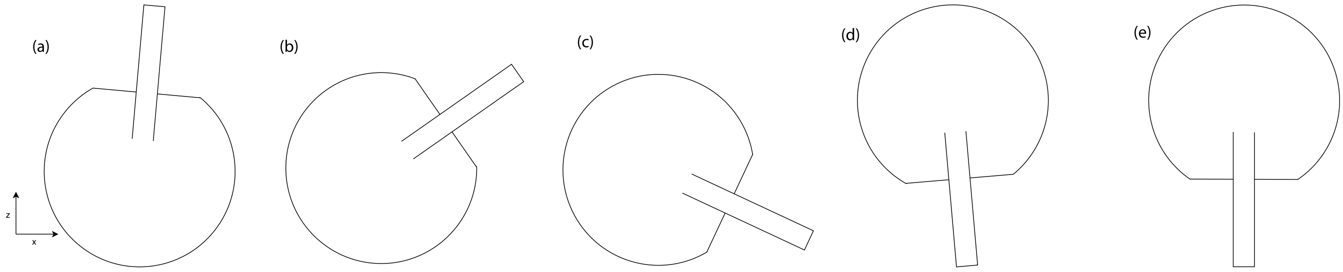


Figure 2. Phases **I** to **V** of the Tippe top's motion, shown in the xz-plane

Let XYZ be the inertial frame, where the surface the top is on is wholly in the XY-plane. The frame xyz is defined as above, and reached from XYZ via rotation around the Z axis by ϕ . The transformation from the XYZ frame to frame xyz is shown in Figure 3(a). In particular, $\hat{\mathbf{z}} = \hat{\mathbf{Z}}$.

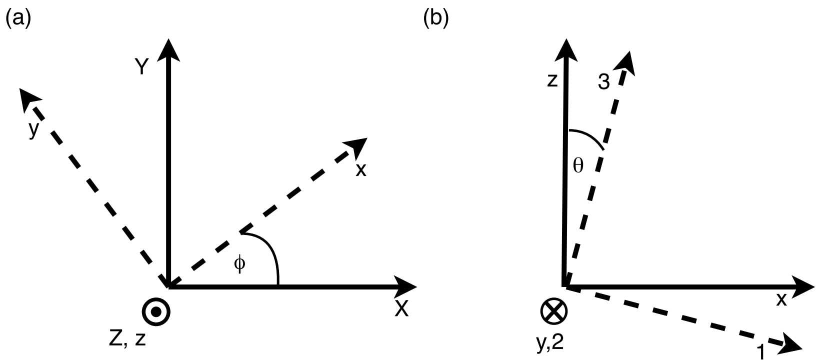


Figure 3. Transformations between reference frames: (a) to xyz from XYZ , and (b) to 123 from xyz

Any rotational motion in 3-dimensional space can be described by the three Euler angles (θ, ϕ, ψ). The transformations between the inertial frame XYZ, the intermediate frame xyz, and the top's frame 123 can be understood in terms of these Euler angles.

In our description of the Tippe top's motion, the angles θ and ϕ are the standard zenith and azimuthal angles respectively, in spherical polar coordinates. In the XYZ frame they are defined as follows: θ is the angle of the top's symmetry axis from the vertical Z-axis, representing how far from vertical its stem is, while ϕ represents the top's angular position about the Z-axis, and is defined as the angle between the XZ-plane and the plane through points $O,\ A,\ C$ (i.e. the vertical projection of the top's symmetry axis).

The third Euler angle ψ describes the rotation of the top about its own symmetry axis, i.e. its 'spin', which has angular velocity $\dot{\psi}$.

The reference frame of the spinning top is defined as a new rotating frame 123, which is reached by rotating xyz by θ around $\hat{\mathbf{y}}$: 'tilting' the $\hat{\mathbf{z}}$-axis down by θ to meet the top's symmetry axis $\hat{\mathbf{3}}$. The transformation from the xyz frame to the 123 frame is shown in Figure 3(b). In particular, $\hat{\mathbf{2}} = \hat{\mathbf{y}}$.

NOTE: For a reference frame $\widetilde{\mathbf{K}}$ rotating in inertial frame $\mathbf{K}$ with angular velocity $\boldsymbol{\omega}$, the time derivatives of a vector $\mathbf{A}$ within both frames $\mathbf{K}$ and $\widetilde{\mathbf{K}}$ are related via:

$$\left(\frac{\partial \mathbf{A}}{\partial t}\right)_{\mathbf{K}} = \left(\frac{\partial \mathbf{A}}{\partial t}\right)_{\widetilde{\mathbf{K}}} + \boldsymbol{\omega} \times \mathbf{A} \quad (1)$$

The motion that a Tippe top undergoes is complex, involving the time evolution of the three Euler angles, as well the translational velocities (or positions) and the motion of the top's symmetry axis. All of these parameters are coupled. To solve for the motion of a Tippe top, one would use standard tools including Newton's laws to prepare the system of equations, then program a computer to solve them numerically via simulation.

In this question, you will perform the first part of this process, investigating the physics of the Tippe top to set up the system of equations.

Friction between the Tippe top and the surface it is moving on drives the motion of the Tippe top. Assume that the top remains in contact with the floor at point A, until such time as the stem contacts the floor. It is in motion at point A with velocity $\mathbf{v}_A$ relative to the floor. The frictional coefficient μ_k between the top and floor is kinetic, with $|\mathbf{F_f}| = \mu_k N$, where $\mathbf{F}_f = F_{f,x}\hat{\mathbf{x}} + F_{f,y}\hat{\mathbf{y}}$ is the frictional force, and N is the magnitude of the normal force. Assume that the top is initially set spinning only, i.e. there is no translational impulse given to the top.

Let the mass of the Tippe top be m. Its moments of inertia are: I_3 about the axis of symmetry is, and $I_1 = I_2$ about the mutually perpendicular principal axes. Let $\mathbf{s}$ be the position vector of the centre of mass, and $\mathbf{a} = \overrightarrow{CA}$ be the vector from the centre of mass to the point of contact.

Unless otherwise specified, give your answers in the xyz reference frame for full marks. All torques and angular momentum are considered about the centre of mass C, unless otherwise specified. You may give your answers in terms of N. Except for part **A.8**, you need only consider the top where $\theta < \frac{\pi}{2}$, and the stem is not in contact with the floor.

A.1	Find the total external force $\mathbf{F}_{\text{ext}}$ on the Tippe top. Draw a free body diagram of the top, projected onto each of the xz- and xy-planes. Indicate the direction of $\mathbf{v}_A$ in the space provided, on your diagram in the xy-plane.	1pt
A.2	Find the total external torque $\boldsymbol{\tau}_{\text{ext}}$ on the Tippe top about the centre of mass.	0.8pt
A.3	Given the contact condition, i.e. $(\mathbf{s} + \mathbf{a}) \cdot \hat{z} = 0$, show that the velocity at A has no component in the z-direction, i.e. we can write $\mathbf{v}_A = v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}}$.	0.4pt
A.4	Find the total angular velocity ω of the rotating top about its centre of mass C in terms of the time derivatives of the Euler angles: $\dot{\theta} = \frac{d\theta}{dt}$, $\dot{\phi} = \frac{d\phi}{dt}$, and $\dot{\psi} = \frac{d\psi}{dt}$. Use Figure 3 if this is helpful. Give your answer in the xyz frame, and in the 123 frame.	0.8pt
A.5	Find the total energy of a spinning Tippe top, in terms of time derivatives of the Euler angles, v_x, and v_y. For partial marks, you may leave your answer in terms of $\dot{\mathbf{s}} = \frac{d\mathbf{s}}{dt}$.	1pt

A.6	Find the rate of change of the angular momentum about the z-axis.	0.4pt

A.7	Which force(s) do work against gravity? Find an expression for the instantaneous rate of change of the top's energy - you may leave your answer in terms of $\mathbf{v}_A$. Identify and identify the components of the force and the torque that cause the change(s) in energy terms in **A.5**.	1.4pt

A.8	Qualitatively sketch the following energy terms in the answer sheet as a function of time, over the top's motion through the five phases **I** to **V** shown in Figure 2: the total energy E_T, gravitational potential energy U_G, translational kinetic energy K_T, and rotational kinetic energy K_R. The energy axes of your sketches are not required to be to scale.	2pt

A.9 Show that the components of the angular momentum **L** and angular velocity $\boldsymbol{\omega}$ that are perpendicular to the $\hat{\mathbf{3}}$ direction are proportional, i.e. (0.5pt)

$$\mathbf{L} \times \hat{\mathbf{3}} = k(\boldsymbol{\omega} \times \hat{\mathbf{3}}), \quad (2)$$

and find the proportionality constant k.

Combining your answers to **A.1** and **A.2** with subsequent results will give you the magnitude N of the normal force, as well as a system of equations, relating the Euler angles, the components v_x and v_y of the velocity at A, the unit vector for the axis of symmetry $\hat{\mathbf{3}}$, and their time derivatives. This system is not integrable, but instead could be solved numerically.

Integrals of motion are quantities which remain constant, and can reduce the dimensionality of the system (i.e. number of simultaneous equations to solve, whether analytically or numerically). Typically quantities such as energy, momentum, and angular momentum are conserved in closed systems, and significantly simplify the problem.

A.10 As you have seen, neither the energy nor the angular momentum are conserved for a Tippe top, due to a dissipative force and external torque. However, there is a related quantity known as Jellett's integral λ, which represents a component of the angular momentum that is conserved, i.e. some vector **v**. such that $\lambda = \mathbf{L} \cdot \mathbf{v}$ is constant in time. (1.7pt)

Use your understanding of the Tippe top and results found to far, to give an expression for such a vector **v**. Show that the time derivative of λ is zero.

Reference sheet for markers

Note: some results below were used for the previous version of part **A.10**, and are no longer needed.

Coordinate systems for convenience (note: use of matrices not needed) xyz from XYZ

$$\begin{bmatrix}\hat{\mathbf{x}}\\ \hat{\mathbf{y}}\\ \hat{\mathbf{z}}\end{bmatrix} = \begin{bmatrix}\cos\phi & \sin\phi & 0\\ -\sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\hat{\mathbf{X}}\\ \hat{\mathbf{Y}}\\ \hat{\mathbf{Z}}\end{bmatrix}$$

123 from xyz

$$\begin{bmatrix}\hat{\mathbf{1}}\\ \hat{\mathbf{2}}\\ \hat{\mathbf{3}}\end{bmatrix} = \begin{bmatrix}\cos\theta & 0 & -\sin\theta\\ 0 & 1 & 0\\ \sin\theta & 0 & \cos\theta\end{bmatrix}\begin{bmatrix}\hat{\mathbf{x}}\\ \hat{\mathbf{y}}\\ \hat{\mathbf{z}}\end{bmatrix}$$

Position of point A from centre of mass, in xyz and 123 frames:

$$\begin{aligned}\mathbf{a} &= \alpha R\hat{\mathbf{3}} - R\hat{\mathbf{z}} \qquad (1)\\ &= \alpha R\sin\theta\hat{\mathbf{x}} + R(\alpha\cos\theta - 1)\hat{\mathbf{z}}\\ &= R\sin\theta\hat{\mathbf{1}} + R(\alpha - \cos\theta)\hat{\mathbf{3}}\end{aligned}$$

Useful products:

$$\hat{\mathbf{z}} \times \hat{\mathbf{3}} = \sin\theta\hat{\mathbf{y}} \qquad (2)$$

$$(3)$$

Note (given in question):

$$\left(\frac{\partial\mathbf{A}}{\partial t}\right)_{\mathbf{K}} = \left(\frac{\partial\mathbf{A}}{\partial t}\right)_{\widetilde{\mathbf{K}}} + \boldsymbol{\omega}\times\mathbf{A} \qquad (4)$$

Time derivatives:

$$\dot{\hat{\mathbf{3}}} = \boldsymbol{\omega}\times\hat{\mathbf{3}} \qquad (5)$$

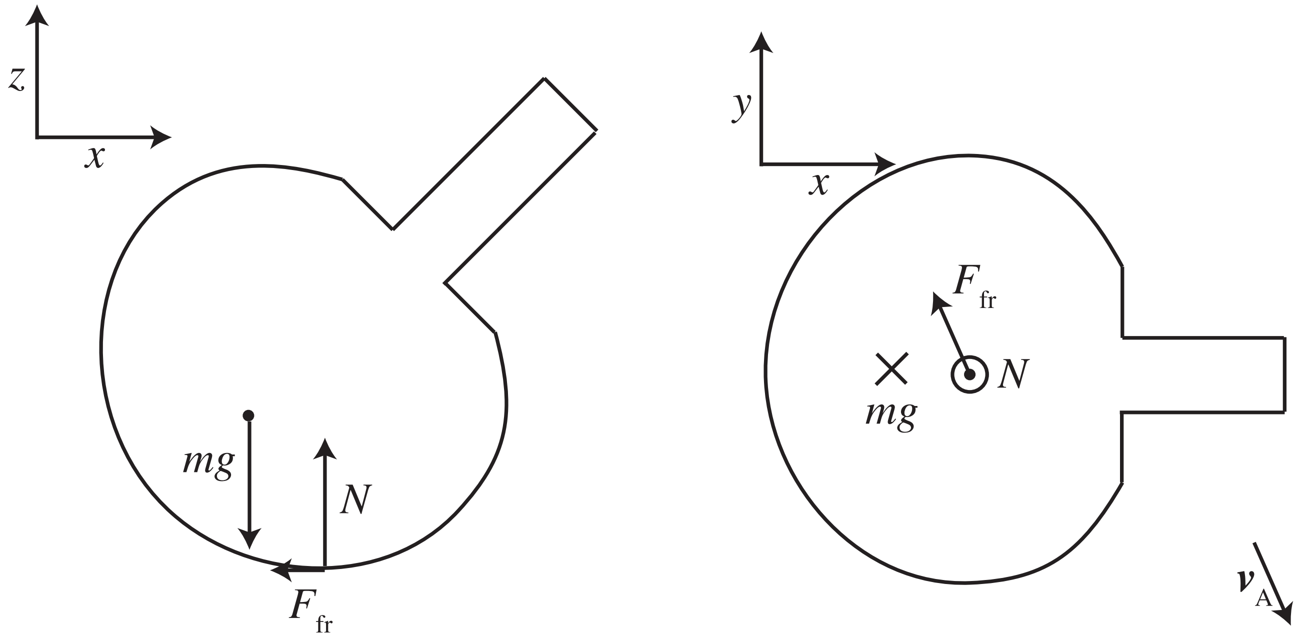
$$\dot{\hat{\mathbf{x}}} = \dot{\phi}\hat{\mathbf{y}} \qquad (6)$$

$$\dot{\hat{\mathbf{y}}} = -\dot{\phi}\hat{\mathbf{x}} \qquad (7)$$

Solutions: Tippe Top

1. (**1.0 marks**)

Free body diagrams:



Note: the direction of $\mathbf{F}_f$ must be opposite to the direction of $\mathbf{v}_A$, but is otherwise unimportant. Sum of forces:

$$\begin{aligned}\mathbf{F}_{\text{ext}} &= (N - mg)\hat{z} + \mathbf{F}_f \quad \text{(sufficient for full marks)} \qquad (8)\\ &= (N - mg)\hat{z} - \frac{\mu_k N}{|v_A|}\mathbf{v_A}\end{aligned}$$

Sketched $\mathbf{v_A}$ must be in opposite direction to $\mathbf{F}_f$ on xy diagram.

2. (**0.8 marks**)

Sum of torques:

$$\begin{aligned}\boldsymbol{\tau}_{\text{ext}} &= \mathbf{a} \times (N\hat{\mathbf{z}} + \mathbf{F}_f) \qquad (9)\\ &= (\alpha R\hat{\mathbf{3}} - R\hat{\mathbf{z}}) \times (N\hat{\mathbf{z}} + F_{f,x}\hat{\mathbf{x}} + F_{f,y}\hat{\mathbf{y}})\\ &= \alpha RN\hat{\mathbf{3}} \times \hat{\mathbf{z}} + \alpha R(\sin\theta\hat{\mathbf{x}} + \cos\theta\hat{\mathbf{z}}) \times (F_{f,x}\hat{\mathbf{x}} + F_{f,y}\hat{\mathbf{y}}) - R\hat{\mathbf{z}} \times (F_{f,x}\hat{\mathbf{x}} + F_{f,y}\hat{\mathbf{y}})\\ &= -\alpha RN\sin\theta\hat{\mathbf{y}} + \alpha R\sin\theta F_{f,y}\hat{\mathbf{z}} + \alpha R\cos\theta F_{f,x}\hat{\mathbf{y}} - \alpha R\cos\theta F_{f,y}\hat{\mathbf{x}} - RF_{f,x}\hat{\mathbf{y}} + RF_{f,x}\hat{\mathbf{x}}\\ &= RF_{f,y}(1 - \alpha\cos\theta)\hat{\mathbf{x}} + [RF_{f,x}(\alpha\cos\theta - 1) - \alpha RN\sin\theta]\,\hat{\mathbf{y}} + \alpha R\sin\theta F_{f,y}\hat{\mathbf{z}} \qquad (10)\end{aligned}$$

3. (**0.4 marks**)

Motion at A satisfies

$$\mathbf{v_A} = \dot{\mathbf{s}} + \boldsymbol{\omega} \times \mathbf{a} \qquad (11)$$

where $\boldsymbol{\omega}$ is the total angular velocity of the top in the centre of mass frame (this is deteremined in the next part). Want to show that $\mathbf{v_A} \cdot \hat{\mathbf{z}} = 0$.

To show this, take time derivative of contact condition in XYZ or xyz frame (note: either is suitable, as we only need the $\hat{\mathbf{z}}$ component, and $\hat{\mathbf{z}} = \hat{\mathbf{Z}}$).

Contact condition:

$$
\begin{aligned}
(\mathbf{s} + \mathbf{a}) \cdot \hat{\mathbf{z}} &= 0 \quad \text{at all times} \\
\Rightarrow \frac{d}{dt} (\mathbf{s} + \mathbf{a}) \cdot \hat{\mathbf{z}} &= 0 \quad \text{at all times}
\end{aligned} \tag{12}
$$

Note we only care about the z-component, and $(\boldsymbol{\omega} \times \hat{\mathbf{z}}) \cdot \hat{\mathbf{z}} = 0$. Then, using 11, 1, and 5,

$$
\begin{aligned}
\mathbf{v_A} \cdot \hat{\mathbf{z}} &= (\dot{\mathbf{s}} + \boldsymbol{\omega} \times \mathbf{a}) \cdot \hat{\mathbf{z}} \\
&= \left(\dot{\mathbf{s}} + \alpha R \boldsymbol{\omega} \times \hat{\mathbf{3}}\right) \cdot \hat{\mathbf{z}} \\
&= \left(\dot{\mathbf{s}} + \alpha R \frac{d\hat{\mathbf{3}}}{dt}\right) \cdot \hat{\mathbf{z}} \\
&= (\dot{\mathbf{s}} + \dot{\mathbf{a}}) \cdot \hat{\mathbf{z}} = 0
\end{aligned} \tag{13}
$$

4. (**0.8 marks**)

Total angular velocity $\boldsymbol{\omega}$ of top is the sum of three distinct rotations:

$$
\boldsymbol{\omega} = \dot{\theta}\hat{\mathbf{2}} + \dot{\phi}\hat{\mathbf{z}} + \dot{\psi}\hat{\mathbf{3}}
$$

Use transformations shown in figure 3 or otherwise to transform into xyz or 123 frame:

$$
\boldsymbol{\omega} = \dot{\psi} \sin\theta \hat{\mathbf{x}} + \dot{\theta}\hat{\mathbf{y}} + (\dot{\psi}\cos\theta + \dot{\phi})\hat{\mathbf{z}} \tag{14}
$$

$$
\boldsymbol{\omega} = -\dot{\phi}\sin\theta \hat{\mathbf{1}} + \dot{\theta}\hat{\mathbf{2}} + (\dot{\psi} + \dot{\phi}\cos\theta)\hat{\mathbf{3}} \tag{15}
$$

5. (**1.0 marks**)

Where $\mathbf{I}$ is the inertia tensor

$$
\begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_1 & 0 \\ 0 & 0 & I_3, \end{bmatrix}
$$

we have

$$
\begin{aligned}
E_T &= K_T + K_R + U_G \\
&= \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{I}\boldsymbol{\omega} + \frac{1}{2} m \dot{\mathbf{s}}^2 + mgR(1 - \alpha\cos\theta)
\end{aligned}
$$

From 11,

$$
\begin{aligned}
\dot{\mathbf{s}} &= \mathbf{v_A} - \boldsymbol{\omega} \times \mathbf{a} \\
&= \mathbf{v_A} - (\dot{\theta}\hat{\mathbf{2}} + \dot{\phi}\hat{\mathbf{z}} + \dot{\psi}\hat{\mathbf{3}}) \times (\alpha R \hat{\mathbf{3}} - R\hat{\mathbf{z}}) \\
&= v_x \hat{\mathbf{x}} + v_y \hat{\mathbf{y}} - \left(\dot{\theta}\alpha R \hat{\mathbf{1}} - \dot{\theta} R \hat{\mathbf{z}} + \dot{\phi}\alpha R \hat{\mathbf{z}} \times \hat{\mathbf{3}} - \dot{\psi} R \hat{\mathbf{3}} \times \hat{\mathbf{z}}\right) \\
&= \left(v_x + \dot{\theta} R (1 - \alpha\cos\theta)\right)\hat{\mathbf{x}} + \left(v_y - R\sin\theta(\alpha\dot{\phi} + \dot{\psi})\right)\hat{\mathbf{y}} + \dot{\theta}\alpha R \sin\theta \hat{\mathbf{z}}
\end{aligned}
$$

using 2. Thus

$$
\begin{aligned}
E_T = &\frac{1}{2}\left[I_1(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2) + I_3(\dot{\psi} + \dot{\phi}\cos\theta)^2\right] \\
&+ \frac{m}{2}\left[\left(v_x + \dot{\theta}R(1 - \alpha\cos\theta)\right)^2 + \left(v_y - R\sin\theta(\alpha\dot{\phi} + \dot{\psi})\right)^2 + \dot{\theta}^2\alpha^2 R^2 \sin^2\theta\right] + mgR(1 - \alpha\cos\theta)
\end{aligned}
$$

6. (**0.4 marks**)

From 10,

$$\frac{d\mathbf{L}}{dt} \cdot \hat{\mathbf{z}} = \sum \boldsymbol{\tau} \cdot \hat{\mathbf{z}} = \alpha R \sin\theta F_{f,y} \tag{16}$$

7. (**1.4 marks**)

Changes in energy: $h = \mathbf{s} \cdot \hat{\mathbf{z}}$ increases, so $\dot{U}_G > 0$.

At start and end (phases **I** and **V**) there is little translation so $K_T \sim 0$ at **I** and **V**. Thus, energy transfer is from K_R to U_G.

Normal force does no work. Frictional force does work at point A. Direction is $-\mathbf{v_A}$:

$$W = \int \mathbf{F}_f \cdot \mathbf{v_A}\ dt < 0$$

$$\Rightarrow \frac{d}{dt} E_T = -\mu_k N |\mathbf{v_A}|$$

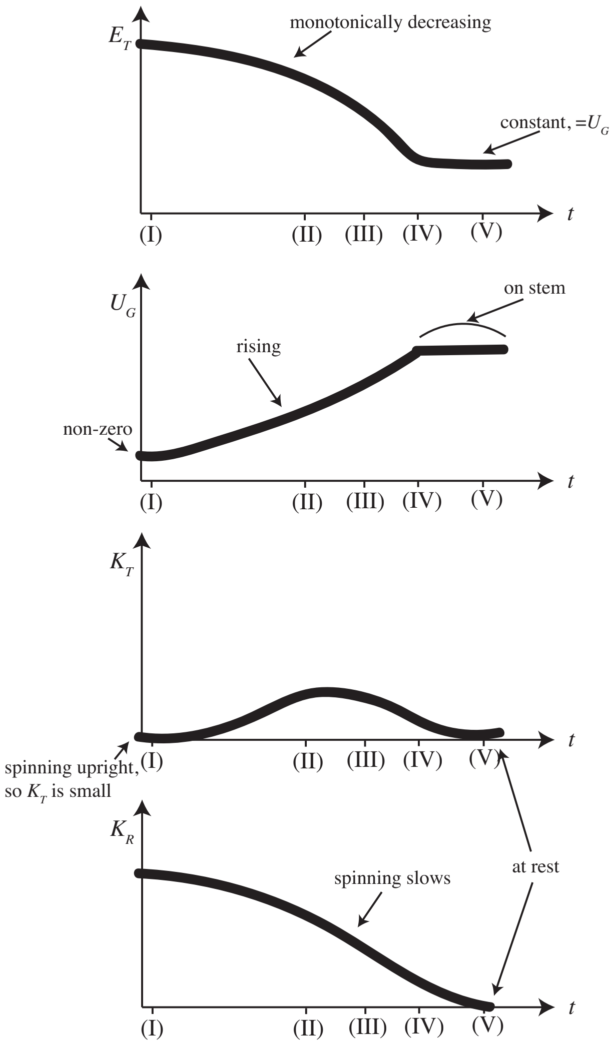
Thus $\mathbf{F}_f$ decreases the total energy monotonically.

16 implies only the $\mathbf{F}_f.\hat{\mathbf{y}}$ acts to decrease $\mathbf{L} \cdot \hat{\mathbf{z}}$. Energy transfer from K_R to U_G, caused by component of frictional force in $\hat{\mathbf{y}}$ direction, so component of resultant torque is in the $\mathbf{a} \times \hat{\mathbf{y}}$ direction.

8. (**2.0 marks**)

Expectation (see figure):

- E_T: monotonically decreasing
- K_R: monotonically decreasing; zero at V
- K_T: zero at I and V; higher between; close to zero at IV
- U_G: flat at start and finish; higher at end; increases from I to IV then flat; increase roughly at same time that K_{rot} decreases



9. (**0.5 marks**)

   From 15,

$$\mathbf{L} = \mathbf{I}\boldsymbol{\omega} = I_1 \left(-\dot{\phi} \sin\theta \hat{\mathbf{1}} + \dot{\theta} \hat{\mathbf{2}} \right) + I_3(\dot{\psi} + \dot{\phi}\cos\theta)\hat{\mathbf{3}} \tag{17}$$

   Taking cross product with $\hat{\mathbf{3}}$:

$$\begin{aligned} \mathbf{L} \times \hat{\mathbf{3}} &= I_1 \left(\dot{\phi} \sin\theta \hat{\mathbf{2}} + \dot{\theta} \hat{\mathbf{1}} \right) \\ &= I_1(\boldsymbol{\omega} \times \hat{\mathbf{3}}) \end{aligned} \tag{18}$$

10. (**1.7 marks**)

   About any axis through the centre of mass,

$$\frac{d\mathbf{L}}{dt} \neq 0 \Leftrightarrow \tau_{\text{ext}} \neq 0$$

   External torque given by 9,

$$\begin{aligned} \boldsymbol{\tau}_{\text{ext}} &= \mathbf{a} \times (N\hat{\mathbf{z}} + \mathbf{F}_f) \\ &\Rightarrow \tau_{\text{ext}} \cdot \mathbf{a} = 0 \\ &\frac{d\mathbf{L}}{dt} \cdot \mathbf{a} = 0 \end{aligned}$$

   Thus, angular momentum in the direction of $\mathbf{a}$ must be constant, so $\mathbf{v} = \mathbf{a}$.

   To demonstrate this mathematically, 5, 10, 18 allow

$$\begin{aligned} -\dot{\lambda} &= \frac{d\mathbf{L}}{dt} \cdot \mathbf{a} + \alpha R \mathbf{L} \cdot \frac{d\hat{3}}{dt} \\ &= (\mathbf{a} \times (N\hat{\mathbf{z}} + \mathbf{F_f})) \cdot \mathbf{a} + \frac{\alpha R}{I_1} \mathbf{L} \cdot (\boldsymbol{\omega} \times \mathbf{L}) \\ &= 0 \end{aligned}$$