QUESTION 3: SOLUTION

1. Using Coulomb's Law, we write the electric field at a distance r is given by

$$E_p = \frac{q}{4\pi\varepsilon_0 (r-a)^2} - \frac{q}{4\pi\varepsilon_0 (r+a)^2}$$

$$E_p = \frac{q}{4\pi\varepsilon_0 r^2}\left(\frac{1}{\left(1-\frac{a}{r}\right)^2} - \frac{1}{\left(1+\frac{a}{r}\right)^2}\right) \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Using binomial expansion for small a,

$$\begin{aligned} E_p &= \frac{q}{4\pi\varepsilon_0 r^2}\left(1+\frac{2a}{r}-1+\frac{2a}{r}\right) \\ &= +\frac{4qa}{4\pi\varepsilon_0 r^3} = +\frac{qa}{\pi\varepsilon_0 r^3} \qquad \ldots\ldots\ldots\ldots\ldots(2) \\ &= \frac{2p}{4\pi\varepsilon_0 r^3} \end{aligned}$$

2. The electric field seen by the atom from the ion is

$$\vec{E}_{ion} = -\frac{Q}{4\pi\varepsilon_0 r^2}\hat{r} \qquad \ldots\ldots\ldots\ldots\ldots (3)$$

The induced dipole moment is then simply

$$\vec{p} = \alpha\vec{E}_{ion} = -\frac{\alpha Q}{4\pi\varepsilon_0 r^2}\hat{r} \qquad \ldots\ldots\ldots\ldots\ldots (4)$$

From eq. (2)

$$\vec{E}_p = \frac{2p}{4\pi\varepsilon_0 r^3}\hat{r}$$

The electric field intensity $\vec{E}_p$ at the position of an ion at that instant is, using eq. (4),

$$\vec{E}_p = \frac{1}{4\pi\varepsilon_0 r^3}\left[-\frac{2\alpha Q}{4\pi\varepsilon_0 r^2}\hat{r}\right] = -\frac{\alpha Q}{8\pi^2\varepsilon_0^2 r^5}\hat{r}$$

The force acting on the ion is

$$\vec{f} = Q\vec{E}_p = -\frac{\alpha Q^2}{8\pi^2\varepsilon_0^2 r^5}\hat{r} \qquad \ldots\ldots\ldots\ldots\ldots (5)$$

The "–" sign implies that this force is attractive and Q^2 implies that the force is attractive regardless of the sign of Q.

3. The potential energy of the ion-atom is given by $U = \int_r^\infty \vec{f}.d\vec{r}$ ……………………………….(6)

Using this, $U = \int_r^\infty \vec{f}.d\vec{r} = -\dfrac{\alpha Q^2}{32\pi^2\varepsilon_0^2 r^4}$ ……………………………………………………………….(7)

[Remark: Students might use the term $-\vec{p}\cdot\vec{E}$ which changes only the factor in front.]

4. At the position $r_{\min}$ we have, according to the Principle of Conservation of Angular Momentum,

$$\begin{aligned} m v_{\max} r_{\min} &= m v_0 b \\ v_{\max} &= v_0 \frac{b}{r_{\min}} \end{aligned} \qquad \text{……………..}\ (8)$$

And according to the Principle of Conservation of Energy:

$$\frac{1}{2} m v_{\max}^2 + \frac{-\alpha Q^2}{32\pi^2\varepsilon_0^2 r_{\min}^4} = \frac{1}{2} m v_0^2 \qquad \text{……………..}\ (9)$$

Eqs.(8) & (9):

$$\left(\frac{b}{r_{\min}}\right)^2 - \frac{\alpha Q^2 \Big/ \frac{1}{2} m v_0^2}{32\pi^2\varepsilon_0^2 b^4}\left(\frac{b}{r_{\min}}\right)^4 = 1$$

$$\left(\frac{r_{\min}}{b}\right)^4 - \left(\frac{r_{\min}}{b}\right)^2 + \frac{\alpha Q^2}{16\pi^2\varepsilon_0^2 m v_0^2 b^4} = 0 \qquad \text{……………..}\ (10)$$

The roots of eq. (10) are:

$$r_{\min} = \frac{b}{\sqrt{2}}\left[1 \pm \sqrt{1 - \frac{\alpha Q^2}{4\pi^2\varepsilon_0^2 m v_0^2 b^4}}\right]^{\frac{1}{2}} \qquad \text{……………..}\ (11)$$

[Note that the equation (8) implies that $r_{\min}$ cannot be zero, unless b is itself zero.]
Since the expression has to be valid at $Q = 0$, which gives

$$r_{\min} = \frac{b}{\sqrt{2}}[1 \pm 1]^{\frac{1}{2}}$$

We have to choose "+" sign to make $r_{\min} = b$

Hence,

$$r_{\min} = \frac{b}{\sqrt{2}}\left[1 + \sqrt{1 - \frac{\alpha Q^2}{4\pi^2\varepsilon_0^2 m v_0^2 b^4}}\right]^{\frac{1}{2}} \qquad \text{…………………………………....}(12)$$

5. A spiral trajectory occurs when (12) is imaginary (because there is no minimum distance of approach).

$r_{\min}$ is real under the condition:

$$1 \geq \frac{\alpha Q^2}{4\pi^2 \varepsilon_0^2 m v_0^2 b^4}$$

$$b \geq b_0 = \left(\frac{\alpha Q^2}{4\pi^2 \varepsilon_0^2 m v_0^2} \right)^{\frac{1}{4}} \qquad \text{……………..}\ (13)$$

For $b < b_0 = \left(\frac{\alpha Q^2}{4\pi^2 \varepsilon_0^2 m v_0^2} \right)^{\frac{1}{4}}$ the ion will collide with the atom.

Hence the atom, as seen by the ion, has a cross-sectional area A,

$$A \; = \; \pi b_0^2 \; = \pi \left(\frac{\alpha Q^2}{4\pi^2 \varepsilon_0^2 m v_0^2} \right)^{\frac{1}{2}} \qquad \text{……………..}\ (14)$$