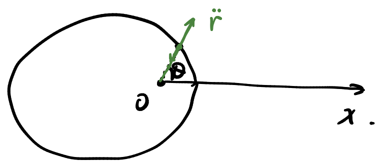
动力学.

有心运动.

例: 开普勒问题(1) 从开一开二, 推导牛顿万有引力定律

普通青年.



开一: $r=\frac{p}{1+e\cos\theta}$ ①

开二: $L=mr^2\dot\theta=$ 常量 ⇒ $\dot\theta=\frac{L}{mr^2}$ ②

牛二: $F=m(\ddot r-r\dot\theta^2)$ ③

①: $\frac{d}{dt}$① ⇒ $\dot r=\frac{-p(e(-\sin\theta))\dot\theta}{(1+e\cos\theta)^2}$

$=\frac{ep\sin\theta\, L}{(1+e\cos\theta)^2\, mr^2}$ ①'

$=\frac{epL\sin\theta}{m p^2}$

$\frac{d}{dt}$①' ⇒ $\ddot r=\frac{eL}{mp}\cos\theta\,\dot\theta$

$=\frac{eL}{mp}\cos\theta\,\frac{L}{mr^2}$

⇒ 代入③: $F/m=\frac{eL^2\cos\theta}{m^2pr^2}-r\frac{L^2}{m^2r^4}$

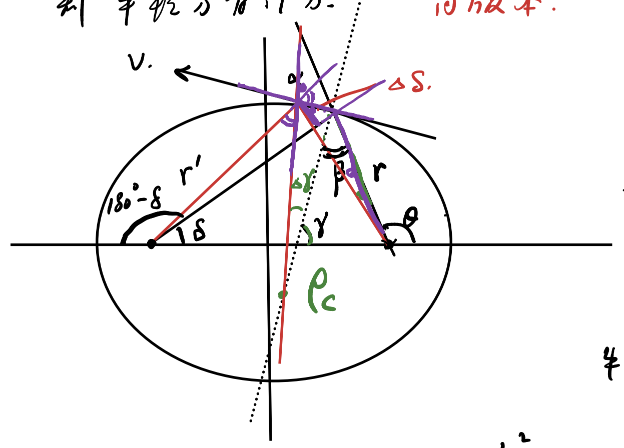
$=\frac{L^2}{m^2r^2}\left(\frac{e\cos\theta}{p}-\frac{1}{r}\right)$

$=\frac{L^2}{m^2r^2}\left(\frac{e\cos\theta}{p}-\frac{1+e\cos\theta}{p}\right)$

$=-\frac{L^2}{m^2p}\cdot\frac{1}{r^2}$

⇒ $F\propto-\frac{1}{r^2}$ ⇒ …

例: 开普勒问题(2) n行方法从开一开二到牛顿万有引力. (简版本.)



β: 用来求任一点曲率半径.

开一: $r=\frac{p}{1+e\cos\theta}$ ①

开二: $m\cdot r\cdot v\cdot\sin\alpha=L$ ②

牛二: $F\cos\beta=\frac{mv^2}{\rho_c}$ ③

② 代入 ③ ⇒ $F\cdot\cos\beta=\frac{m\cdot L^2}{m^2\rho_c r^2\sin^2\alpha}$ $v=\frac{L}{mr\sin\alpha}$

$F=\frac{mL^2}{m^2(\rho_c\cos^3\beta)r^2}$ (*)

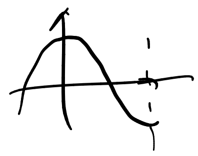
$\rho_c=\frac{p}{\cos^3\beta}$ ← 好用, 目标. $\Delta s=\rho_c\Delta\gamma$.

⇒ $\rho_c=\frac{\Delta s}{\Delta\gamma}$

$\gamma=\delta+\beta=\delta+\frac{\theta-\delta}{2}=\frac{\theta+\delta}{2}$ ⇒ $\Delta\gamma=\frac{\Delta\theta+\Delta\delta}{2}$

$\Delta\theta=\frac{\Delta s\cdot\cos\beta}{r}$, $\Delta\delta=\frac{\Delta s\cdot\cos\beta}{r'}$

$r'=\frac{1}{1+e\cos(180°-\delta)}=\frac{1}{1-e\cos\delta}$



⇒ $\rho_c=\frac{2\Delta s}{\frac{\Delta s\cdot\cos\beta}{r}+\frac{\Delta s\cdot\cos\beta}{r'}}$

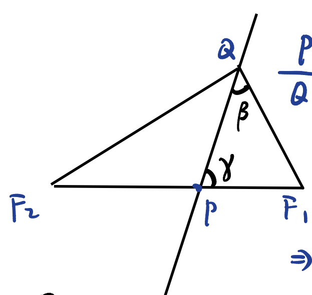
$=\frac{2p}{\cos\beta(1+e\cos\theta+1-e\cos\delta)}$

$=\frac{2p}{\cos\beta(2+e(\cos\theta-\cos\delta))}$

$=\frac{2p}{\cos\beta\left(2-2e\left(\sin\frac{\theta+\delta}{2}\cdot\sin\frac{\theta-\delta}{2}\right)\right)}$

$=\frac{2p}{\cos\beta(2-2e\cdot\sin\gamma\cdot\sin\beta)}$

角平分线定理:



$\frac{PF_1}{QF_1}=\frac{PF_2}{QF_2}=\frac{PF_1+PF_2}{QF_1+QF_2}=e$ ($2c$ / $2a$)

$\frac{\sin\gamma}{QF_1}=\frac{\sin\beta}{PF_1}$

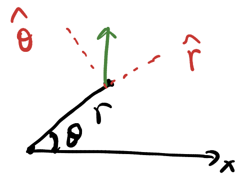
⇒ $\sin\gamma=\frac{QF_1}{PF_1}\sin\beta=\frac{1}{e}\sin\beta$

⇒ $\rho_c=\frac{p}{\cos^3\beta}$ ⇒ 代回 (*) ⇒ 证毕.

例: 开普勒问题(3). 从平方反比力到轨道.

Binet 方程暴力风. $r=\frac{p}{1+e\cos\theta}$

已知: $\vec{F}=-\frac{GMm}{r^2}\hat{r}$. 求证椭圆.



牛二: $\hat\theta$: $0=m(r\ddot\theta+2\dot r\dot\theta)$ ①

$\hat r$: $-\frac{GM}{r^2}=\ddot r-r\dot\theta^2$ ②

① ⇒ $r\frac{d\dot\theta}{dt}=-2\dot\theta\frac{dr}{dt}$ ⇒ $\int\frac{d\dot\theta}{\dot\theta}=-\int 2\frac{dr}{r}$

⇒ $\ln\dot\theta=-2\ln r+C$

⇒ $\dot\theta r^2=常=L/m=l$. ①'

$\ddot r-r\frac{L^2}{m^2r^4}=-\frac{GM}{r^2}=f(r)$

最终: r、θ.

$\dot r=\frac{dr}{dt}=\frac{dr}{d\theta}\frac{d\theta}{dt}=\frac{dr}{d\theta}\frac{L}{mr^2}$

$\ddot r=\frac{d}{d\theta}\left(\frac{dr}{d\theta}\frac{L}{mr^2}\right)\frac{d\theta}{dt}=\frac{d}{d\theta}\left(\frac{dr}{d\theta}\frac{L}{mr^2}\right)\frac{L}{mr^2}$

$\frac{d}{d\theta}\left(\frac{dr}{d\theta}\frac{L}{mr^2}\right)\frac{L}{mr^2}-\frac{L^2}{m^2r^3}=-\frac{GM}{r^2}=f(r)\cdot r^2$

⇒ 令 $r=\frac{1}{u}$, $dr=-\frac{1}{u^2}du$

$\frac{d}{d\theta}\left(-\frac{1}{u^2}\frac{du}{d\theta}\cdot\frac{L}{m}u^2\right)\frac{L}{m}-\frac{L^2}{m^2}u=-GM=f(r)\frac{1}{u^2}$

⇒ $-\frac{d^2u}{d\theta^2}-u=-\frac{GMm^2}{L^2}=f(r)\frac{m^2}{u^2L^2}$ Binet 方程. $f(r)=\frac{F(r)}{m}$.

$\frac{d^2u}{d\theta^2}+\left(u-\frac{GMm^2}{L^2}\right)=0$.

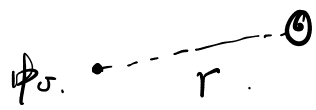
令 $x=u-\frac{GMm^2}{L^2}$ ⇒

$\frac{d^2x}{d\theta^2}+x=0$ ⇒ $x=A\cos(\theta)$

⇒ $\frac{1}{r}-\frac{GMm^2}{L^2}=A\cos\theta$ ⇒ $r=\frac{L^2/GMm^2}{1+e\cos\theta}$.

例: 被扔进回旋中心猪.

$F(r)=-\frac{G'm}{r^3}$



求其轨道 ⇒ $-\frac{d^2u}{d\theta^2}-u=-\frac{GMm^2}{L^2}=f(r)\frac{m^2}{u^2L^2}$

⇒ $-\frac{d^2u}{d\theta^2}-u=-u^3G'\frac{m^2}{u^2L^2}$

⇒ $\frac{d^2u}{d\theta^2}+\left(1-\frac{G'm^2}{L^2}\right)u=0$.

⇒ ① $\frac{G'm^2}{L^2}=1$ ⇒ $\frac{d^2u}{d\theta^2}=0$ ⇒ $u=A+B\theta$

⇒ $r=\frac{1}{A+B\theta}$

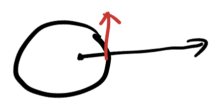
② $1>\frac{G'm^2}{L^2}$ ⇒ $u=A\cos\left(\sqrt{1-\frac{G'm^2}{L^2}}\,\theta\right)$

⇒ $r=\frac{1}{A\cos(\sqrt{\ }\,\theta)}$

③ $1<\frac{G'm^2}{L^2}$ ⇒ $\frac{d^2u}{d\theta^2}-\left(\frac{G'm^2}{L^2}\right)u=0$

$u=e^{\lambda\theta}$ ↗ $\lambda_\pm=\pm\sqrt{\frac{G'm^2}{L^2}}$ ⇒ $u=\frac{e^{+\sqrt{\ }\theta+\varphi}-e^{-\sqrt{\ }\theta+\varphi}}{2}$ +

$u=A\,ch\left(\sqrt{\frac{G'm^2}{L^2}-1}\,\theta+\varphi_0\right)$



⇒ $r=\frac{1}{A\cdot ch(\sqrt{\ }\,\theta+\varphi_0)}$

例: 开普勒问题(4). 从平方反比力 推导轨道.

拉格朗日乡土风 → 有心运动都适用.

已知: $F(r)=-\frac{GMm}{r^2}$. 求证椭圆.

角动量: $L=mr^2\dot\theta$ ⇒ $\dot\theta=\frac{L}{mr^2}$

⇒ $F=-\frac{GMm}{r^2}=m(\ddot r-r\dot\theta^2)$.

⇒ $m\ddot r=mr\dot\theta^2-\frac{GMm}{r^2}$ ↔ $V(r)$

⇒ dr ⇒ $m\frac{d\dot r}{dt}dr=mr\dot\theta^2dr-\frac{GMm}{r^2}dr$

⇒ $m\dot r\,d\dot r=m\frac{L^2}{m^2r^3}dr-\frac{GMm}{r^2}dr$

⇒ $\frac{1}{2}m\dot r^2=-\frac{L^2}{2m}\frac{1}{r^2}+\frac{GMm}{r}+C=-V_{eff}$

$$\Rightarrow \frac{1}{2}m\dot r^2 = -\frac{L^2}{2m}\frac{1}{r^2} + \frac{GMm}{r} + C = -V_{eff}$$

$$\Rightarrow \dot r = \sqrt{\frac{2}{m}\left(C - \frac{L^2}{2mr^2} + \frac{GMm}{r}\right)} = \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{dr}{d\theta}\cdot\frac{d\theta}{dt} \quad \to \frac{dr}{d\theta}$$

$$L = mr^2\dot\theta \Rightarrow \frac{d\theta}{dt} = \frac{L}{mr^2}$$

$$\Rightarrow \int d\theta = \frac{L}{m}\int \frac{dt}{r^2}$$

$$\Rightarrow \int d\theta = \frac{L}{m}\int \frac{dr}{r^2\sqrt{\frac{2}{m}C - \frac{2L^2}{m^2r^2} + \frac{2GMm}{mr}}}$$

$\Rightarrow$ 令 $r = \frac{1}{u} \Rightarrow dr = -\frac{1}{u^2}du$

$$\Rightarrow \int d\theta = \frac{L}{m}\int \frac{-du}{\sqrt{\frac{2}{m}C - \frac{2L^2}{m^2}u^2 + \frac{2GMm}{m}u}}$$

$$-\int \frac{du}{\sigma\sqrt{(1-(u-\sigma)^2)}}$$

令 $u - \sigma = \cos\alpha \Rightarrow du = -\sin\alpha\, d\alpha$

$$+\int \frac{\sin\alpha\, d\alpha}{\sigma\sqrt{\sin^2\alpha}}$$

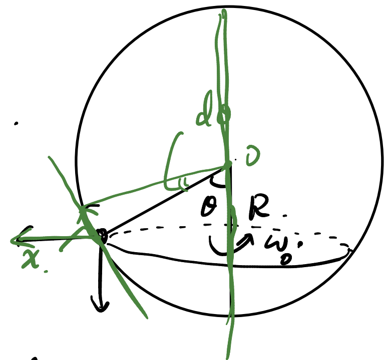
$\theta = \cdots \alpha$

$\Rightarrow u - \sigma = \cos(\theta - \;)$

$\Rightarrow u = \cdots \Rightarrow r = \cdots$

例: 光滑球

求径向小振动的周期.



$\Rightarrow \frac{x}{R} = d\theta \ll 1.$

初: ω_0. θ.

$$\Rightarrow mg\cdot R\sin\theta = m\cdot\omega_0^2\cdot R\sin\theta\cdot R\cos\theta$$

$$\Rightarrow g = \omega_0^2\cdot R\cos\theta \quad ①$$

某时, $\Rightarrow x$. ω. 对O 竖直轴: 角动量守恒,

$$m\cdot\omega_0(R\sin\theta)^2 = m\cdot\omega\cdot(R\sin\theta + x\cos\theta)^2$$

$$\Rightarrow \omega = \left(\frac{R\sin\theta}{R\sin\theta + x\cos\theta}\right)^2\omega_0$$

$\Rightarrow$ 径向运动:

$$m\ddot x = m\omega^2(R\sin\theta + x\cos\theta)\cdot\cos(\theta + d\theta) - mg\sin(\theta + d\theta)$$

$$m\ddot x = mR^4\sin^4\theta\,\omega_0^2(R\sin\theta + x\cos\theta)^{-3}\cdot(\cos\theta - \sin\theta\, d\theta) - mg\cdot(\sin\theta + \cos\theta\cdot d\theta)$$

$$= m\cdot R^4\sin^4\theta\cdot\omega_0^2\cdot\frac{1}{R^3\sin^3\theta}\left(1 - 3\cdot\frac{x\cos\theta}{R\sin\theta}\right)\left(\cos\theta - \sin\theta\frac{x}{R}\right) - mg\cdot(\sin\theta + \cos\theta\cdot d\theta)$$

$$= -3mR\omega_0^2\sin\theta\cdot\frac{x\cos^2\theta}{R\sin\theta} - mR\sin\theta\cdot\omega_0^2\sin\theta\frac{x}{R} - mg\cos\theta\cdot\frac{x}{R}$$

$$= -\left(3m\cdot\omega_0^2\cos^2\theta + m\sin^2\theta\,\omega_0^2 + mg\frac{\sin\theta}{R}\right)\cdot x$$

$$= -(3m\cdot\cos^2\theta + \sin^2\theta + \sin\theta\cos\theta)\,\omega_0^2\cdot x.$$

$m\ddot x = -\sigma\omega_0^2\cdot x \Rightarrow$ 简振

$$\frac{1}{2}m\dot x^2 + \frac{1}{2}k\cdot x^2 = C.$$

$\Downarrow$

$$\omega = \sqrt{\sigma}\cdot\omega_0$$

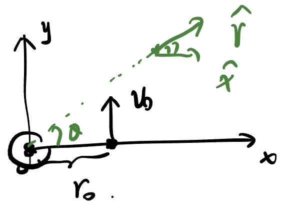
$$\Rightarrow T = \frac{2\pi}{\sqrt{\sigma}\cdot\omega_0} \Rightarrow ✓$$

例. 开普勒问题(5) 从单摆 → 轨道.

拉普拉斯 文艺风

L-R-L 矢量   LL.

定义 $\vec B = \vec v\times\vec L - GMm\cdot\hat r$ ← 单位. 已知: 可 求 椭圆.

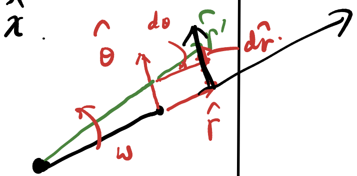


(1) 求 $B_{(0)}$

⇒ $\vec v = v_0\cdot\hat y$. $\vec r = r_0\cdot\hat x$.

$\vec L = m\cdot v_0\cdot r_0\cdot\hat z$. $\vec v\times\vec L = m v_0^2\cdot r_0\cdot\hat x$.

⇒ $\vec B_{(0)} = (m v_0^2 r_0 - GMm)\,\hat x$.



(2) 求证: $\frac{d\vec B}{dt}=0$. $\vec B$ 守恒.   OL.

⇒ $\frac{d\vec B}{dt} = \frac{d\vec v}{dt}\times\vec L + \vec v\times\frac{d\vec L}{dt} - GMm\frac{d\hat r}{dt}$

$= \vec a\times\vec L + 0 - GMm\cdot\hat\theta\cdot d\theta/dt$

其中: $m\vec a = -\frac{GMm}{r^2}\hat r$ $\vec L = m\cdot r^2\dot\theta\,\hat z$

$= -\frac{GM}{r^2}\cdot(-\hat\theta)\cdot m r^2\dot\theta - GMm\,\hat\theta\,\dot\theta$

$= GMm\,\dot\theta\,\hat\theta - GMm\,\dot\theta\,\hat\theta = 0$

(3) 已知 $\vec B$ 守恒. 证: $\hat x$ 求证 椭圆.

⇒ ($\vec B = e\cdot GMm\cdot\hat x$.) $\hat r$ $\hat r\cdot\hat x = \cos\theta$

⇒ $\vec B\cdot\hat r = e\cdot GMm\cdot\cos\theta$

$(\vec v\times\vec L - GMm\cdot\hat r)\cdot\hat r = eGMm\cos\theta$.

⇒ $(\qquad)\cdot\hat r - GMm = eGMm\cos\theta$.

$\vec v\times\vec L = (\dot r\,\hat r + r\dot\theta\,\hat\theta)\times(m\cdot r^2\dot\theta\,\hat z)$

$= (m r^2\dot\theta\,\dot r\,(-\hat\theta) + m r^3\dot\theta^2\,\hat r)$

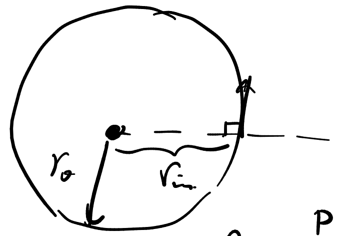
⇒ $(\vec v\times\vec L)\cdot\hat r = m r^3\cdot\dot\theta^2$. $L = m r^2\dot\theta$

⇒ $m r^3\cdot\dot\theta^2 = (1+e\cos\theta)\cdot GMm$

⇒ ~~$m r^3$~~ $\cdot\frac{L^2}{m^2\cdot r^4} = \cdot\cdot$ ⇒ $r = \frac{L^2/GMm^2}{1+e\cos\theta}$

例: 用 L-R-L 矢量. 计算进动. $e\to 0$.

$F = -\frac{GMm}{r^2} - \alpha\frac{GMm}{r_0^3}\cdot r$.



$\alpha \ll 1$.

$r_m = \frac{P}{1+e}$.

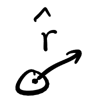
⇒ 圆轨道时: $\omega_0^2\cdot r_0 = \frac{GMm}{r_0^2}(1+\alpha)$

证: $\vec B = \vec v\times\vec L - GMm\,\hat r$ $P = L^2/GMm^2$.

$= (\frac{L}{m\cdot r_m}\cdot L - GMm)\,\hat r$

$= (\frac{L^2(1+e)}{m\cdot P} - GMm)\,\hat r$

$= e\cdot GMm\cdot\hat r$



$\frac{d\vec B}{dt} = \vec a\times\vec L - GMm\cdot\frac{d\hat r}{dt}$

其中 $= -(\frac{GMm}{r^2} + \alpha\cdot\frac{GMm}{r_0^3}\cdot r)\times m r^2\dot\theta(-\hat\theta) - GMm\cdot\dot\theta\,\hat\theta$.

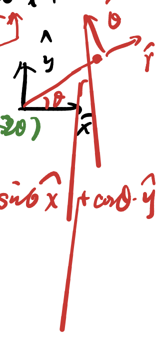
$\frac{d\vec B}{dt} = -\dot\theta\,\alpha\cdot\frac{GMm}{r_0^3}\cdot r^3\,\hat\theta$

$\overline{\frac{d\vec B}{dt}} = \frac{1}{T}\cdot\int_0^T(-\alpha\cdot\frac{GMm}{r_0^3}r^3\cdot\hat\theta\cdot\dot\theta)\,dt$

$= \frac{1}{T}\int_0(-\alpha\cdot\frac{GMm}{r_0^3}\cdot r_0^3(1+e\cos\theta)^3\cdot\hat\theta\,d\theta$

$= \frac{1}{T}\int_0^{2\pi}(-\alpha\cdot GMm\cdot(1+3e\cos\theta)\cdot(-\sin\theta\,\hat x + \cos\theta\,\hat y)\,d\theta$

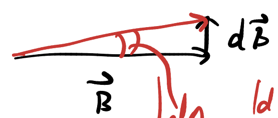
$\frac{d\vec B}{dt} = \frac{1}{T}(-\frac{3}{2}\cdot\alpha\cdot GMm\cdot e)\,\hat y$



$\cos^2\theta = \frac{1}{2}(1+\cos2\theta)$

$\hat\theta = -\sin\theta\,\hat x + \cos\theta\cdot\hat y$.

$\frac{|d\vec B/dt|}{|\vec B|} = -\frac{3}{2}\alpha\cdot\frac{2\pi}{T} = -\frac{3}{2}\alpha\cdot\omega_0$ ✓



$d\theta = \frac{|d\vec B|}{|\vec B|}$

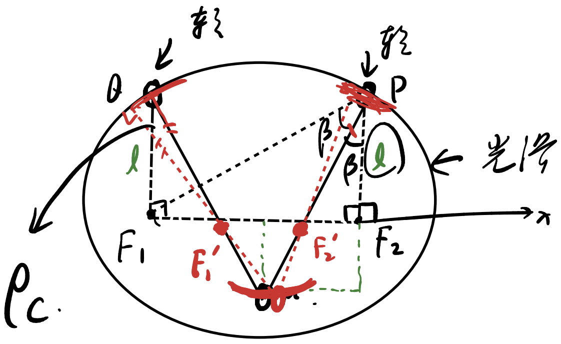
$\Omega = \frac{d\theta}{dt}$

例：套在椭圆上轻质环。

$F_1F_2=\sqrt{3}\,l$

(1) 求绳总长 L

(2) 小振动周期



$\rho=\dfrac{p}{\cos^3\beta}$

$\angle F_1PF_2=60^\circ\Rightarrow\beta=30^\circ$

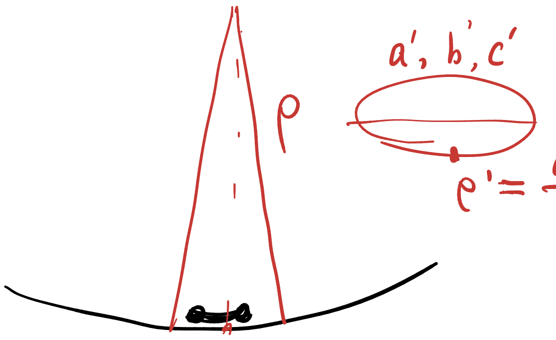
$c=\dfrac{\sqrt{3}}{2}l,\ a=\dfrac{3}{2}l.$

$\Rightarrow L=2\cdot2\cdot c=2\sqrt{3}\,l.$

(2) 下面环在另一椭圆上运动。

$\rho'\leftarrow$

$\Rightarrow T=2\pi\sqrt{\dfrac{\rho'}{g}}$



$\rho'=\dfrac{a}{b'}$

P点处的曲率半径：$\rho_c=\dfrac{p}{\cos^3\beta}$

$\dfrac{p}{1+e\cos\theta}=r$ $\quad\Rightarrow\rho_c=\dfrac{8\cdot l}{3\sqrt{3}}=\dfrac{8}{9}\sqrt{3}\,l.$

当 $\cos\theta=0\Rightarrow p=r=l$

$\Rightarrow 2a'=2\sqrt{3}L-2\rho_c=2\cdot(\frac{1}{9}\sqrt{3})l$

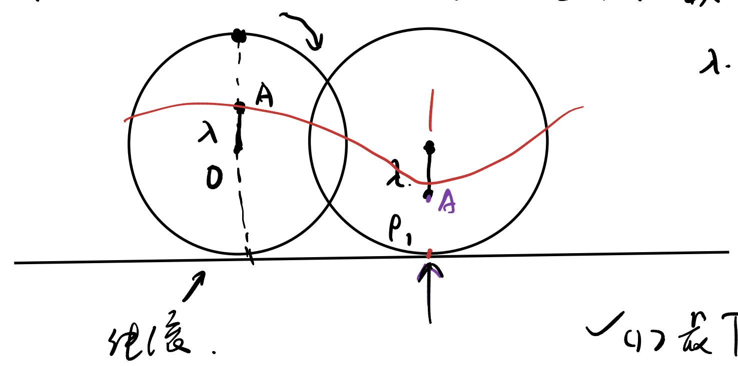
$\Rightarrow a'=\frac{1}{9}\sqrt{3}\,l\quad\Rightarrow b'=\frac{\sqrt{3}}{2}a'=\frac{1}{6}\cdot l$

$\Rightarrow\rho'=\dfrac{a'^2}{b'}=6\cdot\dfrac{3}{9\times9}\cdot l=\dfrac{2}{9}l$

$\Rightarrow T=2\pi\cdot\sqrt{\dfrac{2/9\cdot l}{g}}.$

例：轻质圆板，嵌入质量m点。

$\lambda\cdot r$，r。



✓(1) 最下方时，N_1

(2) OA水平时，N_2，f_2。

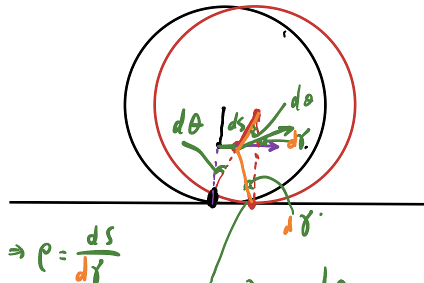
(3) $\mu_{min}=?$

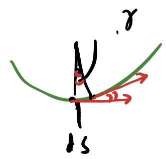
*(4) 不飞，求 λ_{max}。

1)

$\Rightarrow\dfrac{m\cdot v^2}{\rho_1}=N_1-mg.$

速：能：$\frac{1}{2}mv^2=mg\cdot2\lambda\cdot r.$





$\Rightarrow\rho=\dfrac{ds}{d\gamma}.$

$\Rightarrow\rho=\dfrac{ds}{d\gamma}$

$\Rightarrow\lambda r\cdot d\theta=(1-\lambda)r\cdot d\gamma$

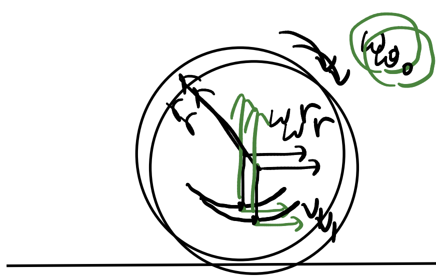
$ds=(1-\lambda)\cdot R\cdot d\theta$

$\Rightarrow\rho=\dfrac{ds}{d\gamma}=\dfrac{(1-\lambda)\cdot r\,d\theta}{\frac{\lambda}{(1-\lambda)}d\theta}=\dfrac{(1-\lambda)^2}{\lambda}\cdot r.$

$N_1=mg+\dfrac{4mg\lambda r}{\frac{(1-\lambda)^2}{\lambda}\cdot r}$

$=\left(1+\dfrac{4\lambda^2}{(1-\lambda)^2}\right)mg$

法二：找另一个简单运动，⇒求出曲率半径。



$v_1=\omega_0r-\omega_0\cdot\lambda r=(1-\lambda)\omega_0r$

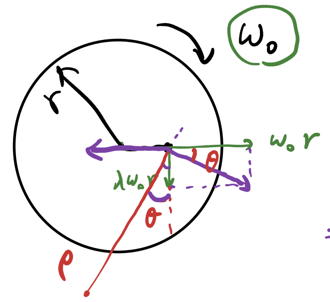
$a_1=\omega_0^2\lambda r$

$\Rightarrow\rho=\dfrac{v_1^2}{a_n}=\dfrac{v_1^2}{a_1}$

$=\dfrac{(1-\lambda)^2\omega_0^2r^2}{\omega_0^2\lambda\cdot r}$

$=\dfrac{(1-\lambda)^2}{\lambda}\cdot r$

(2) 用另一运动求得曲率半径。



$\Rightarrow\tan\theta=\lambda.$

$\sqrt{1+\lambda^2}$, λ, 1

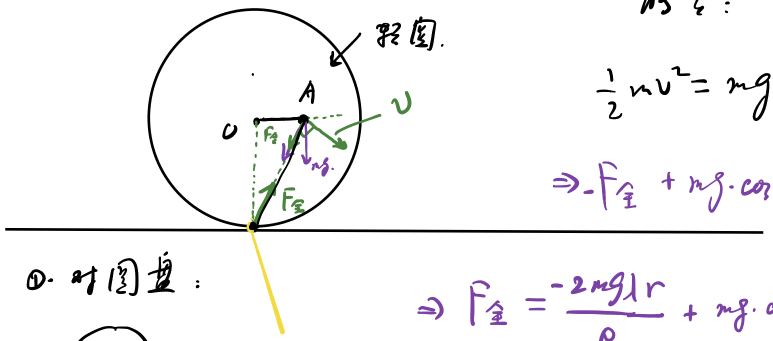
$a_2=\omega_0^2\cdot\lambda r.$

$a_n=a_2\cdot\sin\theta=\dfrac{v^2}{\rho}$

$\Rightarrow\rho=\dfrac{v^2}{a_n}=\dfrac{\omega_0^2r^2+\lambda^2\omega_0^2r^2}{\omega_0^2\lambda r\cdot\frac{\lambda}{\sqrt{1+\lambda^2}}}$

$=\dfrac{(1+\lambda^2)^{3/2}}{\lambda^2}\cdot r.$

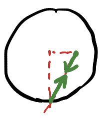
解答：



$\frac{1}{2}mv^2 = mg\cdot\lambda\cdot r$ ①

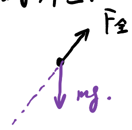
$\Rightarrow -F_全 + mg\cdot\cos\theta = \frac{mv^2}{\rho}$ ②

① 对圆盘：



$\Rightarrow F_全 = \frac{-2mg\lambda r}{\rho} + mg\cdot\cos\theta$

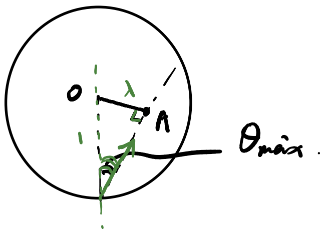
② 对A点：



$N = F_全\cdot\cos\theta$, $f = F_全\cdot\sin\theta$ $\Rightarrow N = \frac{-\lambda^2\cdot 2mg\lambda r}{(1+\lambda^2)^{3/2}\cdot r} + mg\cdot\frac{1}{(1+\lambda^2)^{\frac{1}{2}}}$

$= mg\cdot\left(\frac{-2\lambda^3+(1+\lambda^2)}{(1+\lambda^2)^{3/2}}\right)$ ✓

(3)



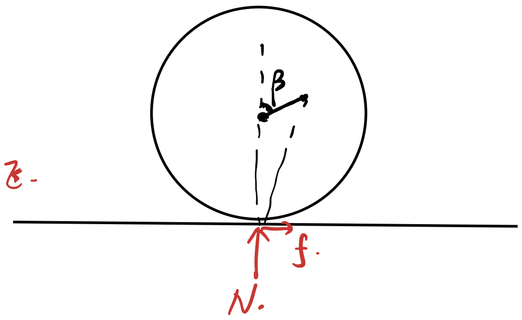
$\Rightarrow \varphi_{min} = \theta_{max}$

$\Rightarrow \mu_{min} = \tan\theta_{max}$

$= \frac{\lambda}{\sqrt{1-\lambda^2}}$

(4) 不飞：

$N > 0$, $f > 0$ } $N=0$, $f=0$ } ⇒ 飞.



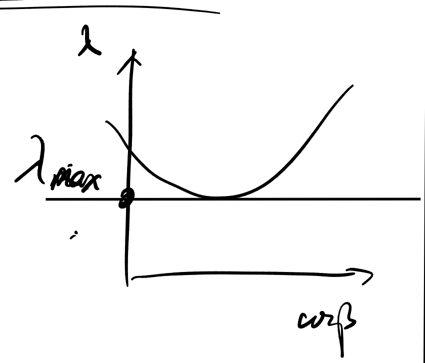
① 先用另一运动 求得任一处的曲率半径. ✓

② 力 牛二 $\Rightarrow N\cdot(\cos\beta, \lambda) = 0$

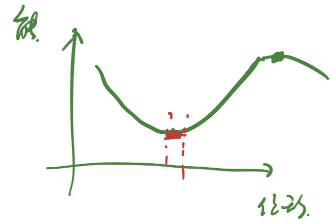
③ 取 $N=0$ ⇒ $\cos\beta$ 是一个合理值 ⇒ 飞.

$(1+\lambda^2-2\lambda\cdot\cos\beta)(1-\lambda\cos\beta) - 2\lambda^2(1+\cos\beta)(\lambda-\cos\beta) = 0$

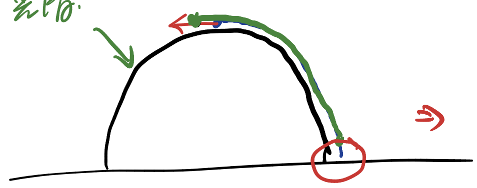
$\lambda_{max} = \frac{7}{9}$



单自由度. 解答. 虚功.

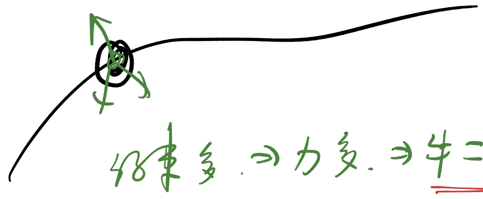


例：光滑



$\Rightarrow F\cdot\delta l + \delta W_G = 0 \Rightarrow F$

当约束多. 自由度少. ⇒ 虚功 (理论力学)



q_1

约束多 ⇒ 力多 ⇒ 牛二 复杂

给一个虚位移 δl ⇒ $\sum \delta W_i = 0$

例：七根火柴棍.

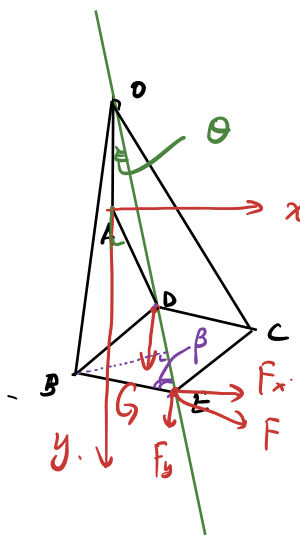
OA 固定. a, b.

$2\times4-7$ 个 ⇒ 1个自由度.

$\theta \to \theta+\delta\theta$

$\Rightarrow \delta W_G + \delta W_F = 0$

δy_D ✓



δx_E ✓ δy_E ✓

$\Rightarrow y_D = a\cdot\cos 2\theta \Rightarrow \delta y_D = a\cdot(-\sin 2\theta)\cdot 2\delta\theta$ ①

$\delta W_G = G\cdot\delta y_D$

$= -2a\sin 2\theta\cdot G\cdot\delta\theta$

$x_E = (OD+DE)\cdot\sin\theta$

$= (2a\cdot\cos\theta + 2a\cdot\cos\beta)\cdot\sin\theta$

△OBD中. 余弦: $a^2 + (2a\cos\theta)^2 + 2a\cdot 2a\cdot\cos\theta\cdot\cos\beta = b^2$

$\Rightarrow \cos\beta = \frac{b^2-a^2-4a^2\cos^2\theta}{4a^2\cdot\cos\theta}$

$= \frac{4a^2\cos^2\theta + b^2 - a^2 - 4a^2\cos^2\theta}{2a\cdot\cos\theta}\cdot\sin\theta$

$x_E = \frac{b^2-a^2}{2a\cos\theta}\cdot\sin\theta$ （$\overline{OE}$）

$\Rightarrow y_E = \overline{OE}\cdot\cos\theta$

$y_E = \frac{b^2-a^2}{2a}$, $\delta\theta$ 无关

$F_y = 0 \Rightarrow F_{min}$.

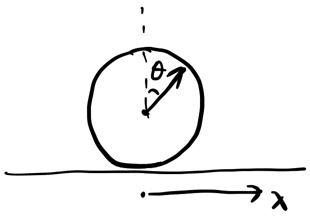
$\Rightarrow \delta x_E = \frac{b^2-a^2}{2a\cos^2\theta}\cdot\delta\theta$

$\Rightarrow \delta W_F = F_x\cdot\frac{b^2-a^2}{2a\cos^2\theta}\cdot\delta\theta$

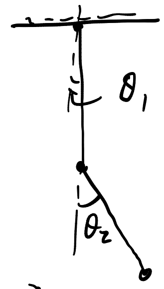
$\Rightarrow \delta W_G + \delta W_F = 0 \Rightarrow F_x = \dots$

多自由度：q_1, q_2 ⋯ ⇒ δq_1, δq_2
相互独立. ⇒ $\sum \delta W_i = 0$.

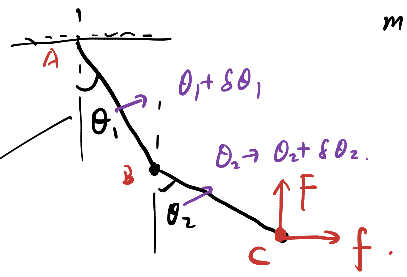
例：$\theta \sim x$



例：



例：m, l. 自由杆.



法一. 力矩. ✓
法二. 虚功/能量

$(5F-1)\cdot\delta x_1 + (3F+2)\cdot\delta y_1 = 0$

$\sum \delta W_i = 0$

$\circlearrowleft(\delta\theta_1) + \circlearrowleft(\delta\theta_2) = 0$.

$\Rightarrow \circlearrowleft(\delta\theta_1) = 0$. $\quad \circlearrowleft(\delta\theta_2) = 0$.

$\delta W_G = -\delta E_{p重} \Rightarrow \delta\left(E_{p重} = -mg\cdot\frac{l}{2}\cos\theta_1 + \cdots\right)$ ①

$\delta W_F \Rightarrow y_c = \cdots \Rightarrow \delta y_c = \cdots$

$\delta W_F = F\cdot\delta y_c$. ② $\quad \delta W_f = \cdots$ ③

① + ② + ③ = 0 ⇒ $F = \cdots$ $\quad f = \cdots$

- 些相互独立广义坐标：x, y, z, θ, I, q.

$q_1, q_2 \cdots q_i$

$T = E_k$. $\quad V = E_p$. 设 $L = T - V$.

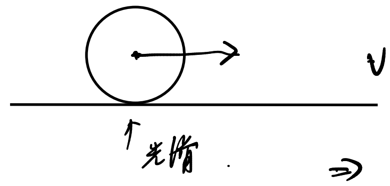
$\frac{1}{2}mv^2 \to \frac{1}{2}m\dot{q}_1^2 \cdots$ $\quad L(\dot{q}_1, \dot{q}_2, \cdots; q_1, q_2 \cdots)$

$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$. ⇐ 拉朗方程 ⇔ 牛二.

$L(\dot{q}_1, \dot{q}_2; q_1) \Rightarrow \frac{\partial L}{\partial q_2} = 0$.

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0$ $\quad \Downarrow \quad \frac{\partial L}{\partial \dot{q}_i} = $ 守恒量.

例：



$T = E_k = \frac{1}{2}m\dot{x}^2$

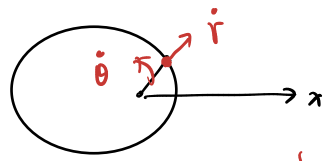
$V = E_p = $ 常.

$\Rightarrow L = T - V = \frac{1}{2}m\dot{x}^2 - C$.

$\Rightarrow \frac{\partial L}{\partial x} = 0 \Rightarrow \frac{\partial L}{\partial \dot{x}} = $ 守恒：$m\dot{x} = P_x$.

一个对称性 ⇒ 守恒量.

例：



$\Rightarrow T = \frac{1}{2}m\left(\dot{r}^2 + (r\dot{\theta})^2\right)$

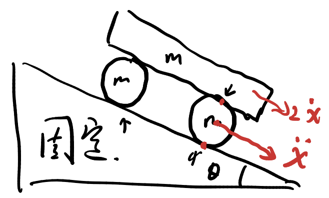
$V = -\frac{GMm}{r}$.

$\Rightarrow L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + \frac{GMm}{r}$

$\Rightarrow \frac{\partial L}{\partial \theta} = 0 \Rightarrow \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} = $ 守恒. $= L$.

旋转角度 ⇒ 角动量守恒.

例：



纯滚. 柱. r, $I = \frac{1}{2}mr^2$
求 $\ddot{x}$.

$E_k = 2\times\left[\frac{1}{2}m\cdot\dot{x}^2 + \frac{1}{2}\cdot\left(\frac{1}{2}mr^2\right)\left(\frac{\dot{x}}{r}\right)^2\right] + \frac{1}{2}m\cdot(2\dot{x})^2$

$= \frac{1}{2}\cdot(3m)\dot{x}^2 + \frac{1}{2}(4m)\dot{x}^2 = \frac{1}{2}(7m)\dot{x}^2$.

$E_p = -2mg\cdot x\cdot\cos\theta - 2\cdot mg\cdot x\cdot\cos\theta = -4mg\,x\cos\theta$

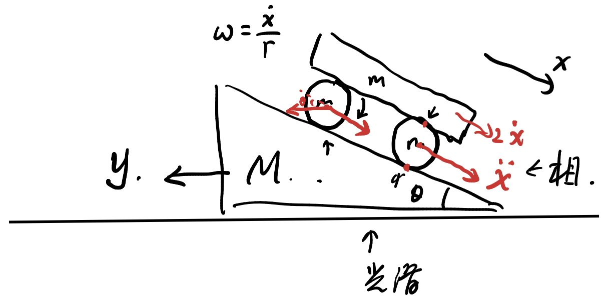
$\Rightarrow a = \ddot{x} = \frac{4}{7}g\cos\theta$.

$E_k + E_p = C \Rightarrow \frac{1}{2}(\circlearrowleft)\cdot\dot{x}^2 + \mathcal{S}\cdot x = 0$.

$\Rightarrow \frac{d}{dt} \Rightarrow \circlearrowleft\cdot\dot{x}\ddot{x} + \mathcal{S}\dot{x} = 0$

$\Rightarrow \ddot{x} = -\frac{\mathcal{S}}{\circlearrowleft}$

例:



$$E_k = \frac{1}{2}M\dot{y}^2$$
$$+\ 2\times\frac{1}{2}m\cdot(\dot{y}^2+\dot{x}^2-2\dot{x}\dot{y}\cdot\cos\theta)$$
$$+\ 2\times\frac{1}{2}(\frac{1}{2}mr^2)\cdot(\frac{\dot{x}}{r})^2$$
$$+\ \frac{1}{2}\cdot m\cdot(\dot{y}^2+(2\dot{x})^2-2\times2\dot{x}\dot{y}\cdot\cos\theta)$$
$$=\frac{1}{2}(M+3m)\dot{y}^2+\frac{1}{2}(7m)\cdot\dot{x}^2-\frac{1}{2}(8m)\dot{x}\dot{y}\cos\theta.$$

$$E_p = -4mg\cdot x\cdot\cos\theta.$$

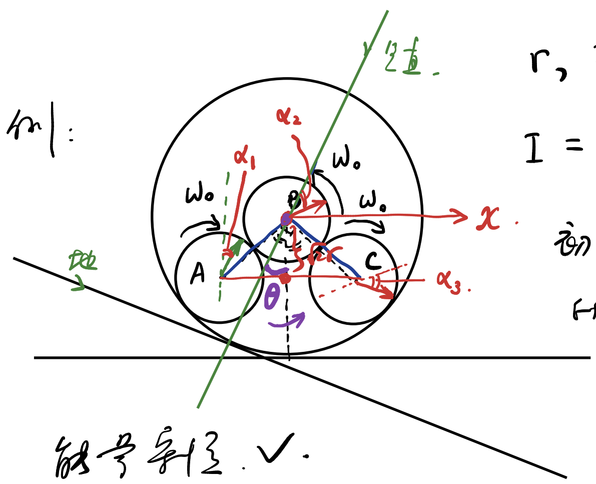
$\Rightarrow L = T-V = E_k - E_p$

$\Rightarrow \frac{\partial L}{\partial y}=0.\ \Rightarrow\ \frac{\partial L}{\partial\dot{y}}=0$

$\Rightarrow \frac{1}{2}\times2(M+3m)\dot{y}-\frac{1}{2}(8m)\cdot\dot{x}\cos\theta = 常量 = P_y.$

水平动量守恒. ⇒ ...

例:



r, 3r, m. 空心筒

$I = mr^2,\ I_大 = m(3r)^2$

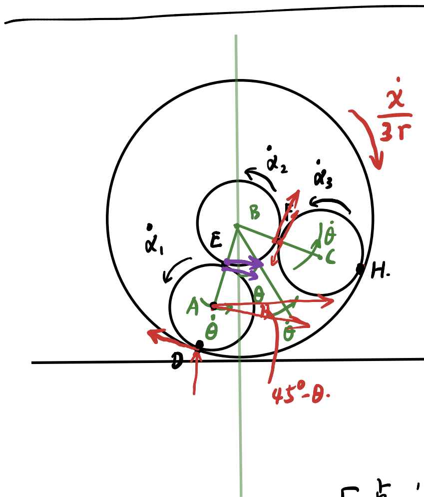
初: A.B.C: W_0, 大筒 $w=0$.

所有纯滚, 求最多偏离竖直方向多远.

能量守恒 ✓.

x、θ、α_1 α_2 α_3. 大筒转动: $\dot{x}/3r$.

在大筒中心平动系中看.



D点速度约束:

A转速 加对A

$\frac{\dot{x}}{3r}\cdot3r = -(2r\cdot\dot{\theta}+r\cdot\dot{\alpha}_1)$ ①.

E点速度:

$r\cdot\dot{\alpha}_2 = 2r\dot{\theta}-r\cdot\dot{\alpha}_1$ ②.

F点速度关联:

$r\dot{\alpha}_2 = 2r\dot{\theta}-r\cdot\dot{\alpha}_3$ ③.

$\Rightarrow \dot{\alpha}_1=\dot{\alpha}_3$

① ⇒ $\dot{\alpha}_1 = \frac{\dot{x}}{r}-2\dot{\theta} = \dot{\alpha}_3$

② ⇒ $\dot{\alpha}_2 = 4\dot{\theta}-\frac{\dot{x}}{r}$

① ⇒ $\dot{\alpha}_1 = \frac{\dot{x}}{r}-2\dot{\theta} = \dot{\alpha}_3$

② ⇒ $\dot{\alpha}_2 = 4\dot{\theta}-\frac{\dot{x}}{r}$

$$\Rightarrow E_p = mg\cdot2R[(\cos45°-\cos(\theta+45°))+(\cos45°-\cos(45°-\theta))]$$
$$= mg\cdot2R\cdot[\cos45°\times2-(\cos(45°+\theta)+\cos(45°-\theta))]$$
$$= mg\cdot2R\cdot[\sqrt{2}-2\cdot\cos45°\cdot\cos\theta]$$
$$E_p = mg\cdot2R\cdot\sqrt{2}[1-\cos\theta].$$

$$E_k = E_大 + E_A + E_B + E_C.$$
$$= \frac{1}{2}m\cdot\dot{x}^2+\frac{1}{2}\cdot(m(3r)^2)(\frac{\dot{x}}{3r})^2$$
$$+\frac{1}{2}m\cdot[\dot{x}^2+(2r\dot{\theta})^2+2\dot{x}(2r\dot{\theta})\cdot\cos(45°-\theta)] \leftarrow E_{kA质}$$
$$+\frac{1}{2}(mr^2)\cdot((\frac{\dot{x}}{r})^2+4\dot{\theta}^2-4\frac{\dot{x}\dot{\theta}}{r}). \leftarrow E_{kAr}$$
$$+\frac{1}{2}\cdot m\cdot\dot{x}^2+\frac{1}{2}(m\cdot r^2)\cdot(16\dot{\theta}^2+\frac{\dot{x}^2}{r^2}-8\frac{\dot{\theta}\dot{x}}{r}) \leftarrow B.$$
$$+\frac{1}{2}m\cdot[\dot{x}^2+(2r\dot{\theta})^2+2\dot{x}(2r\dot{\theta})\cdot\cos(45°+\theta)]$$
$$+\frac{1}{2}(mr^2)\cdot((\frac{\dot{x}}{r})^2+4\dot{\theta}^2-4\frac{\dot{x}\dot{\theta}}{r}). \leftarrow C.$$
$$=\frac{1}{2}Ⓐ\dot{x}^2+\frac{1}{2}Ⓑ\dot{\theta}^2+\frac{1}{2}Ⓒ\cdot\dot{x}\dot{\theta}$$

$\Rightarrow L = T-V = \cdots$ $\frac{\partial L}{\partial x}=0.$

$\frac{\partial L}{\partial\dot{x}} = 守恒量.$

$\frac{\partial L}{\partial\dot{x}} = Ⓐ\dot{x}+\frac{1}{2}Ⓒ\cdot\dot{\theta} = 守恒量.$ ②.

最多偏: $\dot{\theta}=0.$

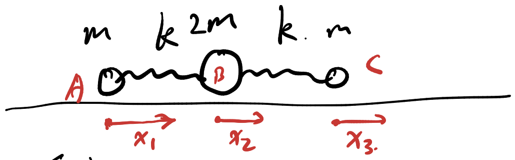
$\dot{\theta}=0$, $\dot{x}$, θ. ⇒ θ ⇒ ✓.

$\dot{x}=?$ ⇒ ✓.

简振模：多个耦合振.（2个）.

2个：$x_1 = A_1\cos(\omega_1 t+\varphi_1)+A_2\cos(\omega_2 t+\varphi_2)$.

例：



质心系中.

⇒ $m x_1 + 2m x_2 + m x_3 = 0$.

⇒ $x_2 = -\frac{1}{2}(x_1+x_3)$

⇒ A：牛二：$m\ddot{x}_1 = k\cdot(x_2-x_1) = -k(\frac{3}{2}x_1+\frac{1}{2}x_3)$ ①.

C：牛二：$m\ddot{x}_3 = -k(x_3-x_2) = -k(\frac{1}{2}x_1+\frac{3}{2}x_3)$ ②

找：ξ, η. ⇒ $\begin{cases}\sigma\ddot{\xi} = -\sigma'\xi \Rightarrow \text{谐振}\\ \sigma\ddot{\eta} = -\sigma'\eta \Rightarrow \text{谐振}\end{cases}$

⇒ $\begin{cases}\xi = A_1\cos(\sigma t+\varphi_1)\\ \eta = A_2\cos(\sigma t+\varphi_2)\end{cases}$

令 $\begin{pmatrix}x_1\\x_3\end{pmatrix} = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}$.

$x_1+x_3=0$　　$x_1=x_3 \Rightarrow x_1-x_3=0$

猜 $\begin{cases}\xi = \frac{1}{\sqrt{2}}(x_1+x_3)\\ \eta = \frac{1}{\sqrt{2}}(x_1-x_3)\end{cases}$

①+② ⇒ $m(\ddot{x}_1+\ddot{x}_3) = -2k\cdot(x_1+x_3)$

$m\ddot{\xi} = -2k\cdot\xi$ ⇒ 谐振.

①−② ⇒ $m\ddot{\eta} = -k\eta$ ⇒ 谐振

⇒ $\begin{cases}x_1 = \frac{1}{\sqrt{2}}(\xi+\eta) = \cdots\\ x_3 = \frac{1}{\sqrt{2}}(\xi-\eta) = \cdots\end{cases}$ ← 初条件.

法一：牛二：$f(\ddot{x}_1, \ddot{x}_2, x_1, x_2) = 0$ ①

$g(\ddot{x}_1, \ddot{x}_2, x_1, x_2) = 0$ ②

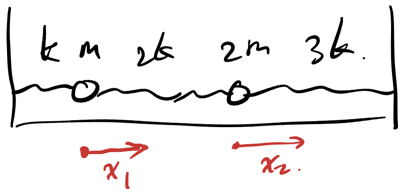
⇒ $f + \lambda(g)$ ⇒

$\sigma\ddot{x}_1 + \tilde{\sigma}\ddot{x}_2 = -(\sigma' x_1 + \tilde{\sigma}' x_2)$

要求 $\frac{\sigma}{\sigma'} = \frac{\tilde{\sigma}}{\tilde{\sigma}'}$ ⇒ λ值.

$\sigma\ddot{\xi} = -\sigma'\xi$ ⇒ 谐振

例：



⇒ $m\ddot{x}_1 = -kx_1 + 2k\cdot(x_2-x_1) = -3kx_1+2kx_2$ ①

$m\ddot{x}_2 = -2k(x_2-x_1) - 3k\cdot x_2 = 2kx_1 - 5kx_2$ ②.

⇒ ①+λ②

⇒ $m(\ddot{x}_1+\lambda\ddot{x}_2) = -((3k-2\lambda k)x_1 + (-2k+5\lambda k)x_2)$

⇒ $\frac{1}{3-2\lambda} = \frac{\lambda}{-2+5\lambda}$

⇒ $-2+5\lambda = 3\lambda - 2\lambda^2$.

⇒ $2\lambda^2+2\lambda-2=0$ ⇒ $\lambda = \frac{-1\pm\sqrt{5}}{2}$

$\xi = (x_1 + \frac{-1+\sqrt{5}}{2}x_2)$ ⇒ ω_1

$\eta = (x_1 + \frac{-1-\sqrt{5}}{2}x_2)$ ⇒ ω_2.

⇒ $x_1 = \cdots$　　$x_2 = \cdots$ ✓.

能量法解振动

单自由度: $E_{总} = E_k + E_p = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = 常$

$\omega = \sqrt{\frac{k}{m}}$

$\Rightarrow \frac{d}{dt} \Rightarrow m\dot{x}\ddot{x} + k\cdot x\cdot\dot{x} = 0 \Rightarrow \ddot{x} + \frac{k}{m}\cdot x = 0$

⇓ 谐振

2个自由度. x_1, x_2

$$\Rightarrow \begin{cases} E_p = \bigcirc x_1^2 + \bigcirc x_2^2 + \bigcirc x_1x_2 \\ E_k = \bigcirc \dot{x}_1^2 + \bigcirc \dot{x}_2^2 + \bigcirc \dot{x}_1\dot{x}_2 \end{cases}$$

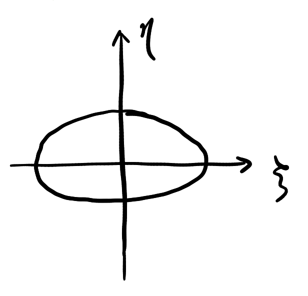
目标: $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix}$

$$\begin{cases} E_p = \bigcirc \xi^2 + \bigcirc \eta^2 \\ E_k = \bigcirc \dot{\xi}^2 + \bigcirc \dot{\eta}^2 \end{cases}$$

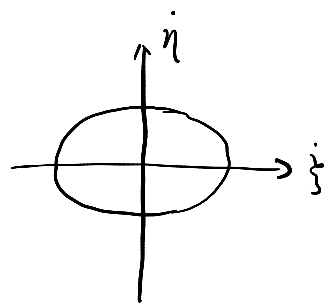
$\bigcirc\xi^2 + \bigcirc\eta^2 + \bigcirc\dot{\xi}^2 + \bigcirc\dot{\eta}^2 = E = 常$

$\frac{d}{dt}$: $2\bigcirc\xi\dot{\xi} + 2\bigcirc\eta\dot{\eta} + 2\bigcirc\dot{\xi}\ddot{\xi} + 2\bigcirc\dot{\eta}\ddot{\eta} = 0$ (grouped: $=0$, $=0$)

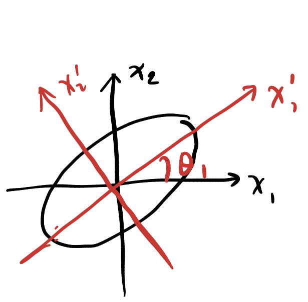
$$\Rightarrow \begin{cases} \bigcirc\xi + \bigcirc\ddot{\xi} = 0 \Rightarrow \\ \bigcirc\eta + \bigcirc\ddot{\eta} = 0 \Rightarrow \end{cases} 2个谐振.$$

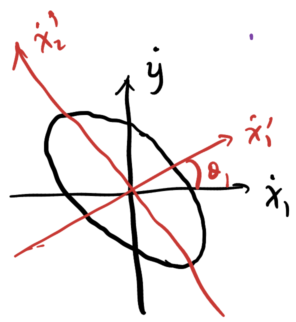


目标.



初始:





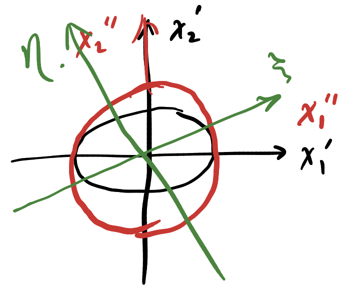
① 先让位形空间变正椭圆: $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{pmatrix}\begin{pmatrix} x_1' \\ x_2' \end{pmatrix}$.

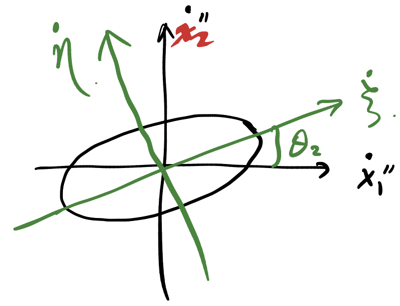
② 再让位形空间变正圆.

$x_2'' = \lambda\cdot x_2'$

$\Rightarrow$ 圆.

②

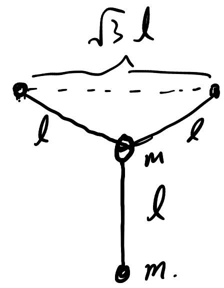




③. 令: $\begin{pmatrix} x_1'' \\ x_2'' \end{pmatrix} = \begin{pmatrix} \cos\theta_2 & \sin\theta_2 \\ -\sin\theta_2 & \cos\theta_2 \end{pmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix}$.

$$\Rightarrow \begin{cases} E_k = \frac{1}{2}\bigcirc\dot{\xi}^2 + \frac{1}{2}\bigcirc\dot{\eta}^2 \\ E_p = \frac{1}{2}\bigcirc\xi^2 + \frac{1}{2}\bigcirc\eta^2 \end{cases} \Rightarrow \surd\ 2个谐振$$

例:



求简正模. 小振动. 光滑.