Methods of Mathematical Physics

Solution of Exercise Problems

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Contents

This is a solution manual of selected exercise problems from *Methods of Mathematical Physics*, 2nd Edition (in Chinese), by Wu Chong-Shi (Peking University Press, Beijing, 2003).

1 Complex Numbers and Complex Functions

Exercises are omitted since they are straightforward.

2 Analytic Functions

2.1 Exercises in the text

2.1.

Proof.
$$i\frac{\partial f}{\partial x} = i\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) = -\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x},\ \frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}.$$

So $i\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y}$ if and only if Cauchy-Riemann equations hold. □

2.2.

Proof. Since $x = \frac{1}{2}(z+z^*)$ and $y = -\frac{i}{2}(z-z^*)$, we have $\frac{\partial x}{\partial z^*} = \frac{1}{2}$ and $\frac{\partial y}{\partial z^*} = \frac{i}{2}$. So

$$\frac{\partial f}{\partial z^*} = \left(\frac{1}{2}\frac{\partial u}{\partial x} + \frac{i}{2}\frac{\partial u}{\partial y}\right) + i\left(\frac{1}{2}\frac{\partial v}{\partial x} + \frac{i}{2}\frac{\partial v}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right).$$

Therefore $\frac{\partial f}{\partial z^*} = 0$ if and only if Cauchy-Riemann equations hold. □

2.3.

Proof. Apply Cauchy-Riemann equations. □

2.4.

Proof. Omitted since the proofs are based on the definition and are similar to those for functions of real variables. □

2.5.

Proof. Let z_1 and z_2 be any two distinct points in the complex plane. Define $f(z) = \exp\left(i\frac{2z-(z_1+z_2)}{z_2-z_1}\pi\right)$. Then $f(z)$ is analytic with $f(z_1) = f(z_2) = -1$. But $f'(z) = \frac{2i\pi}{z_2-z_1}\exp\left(i\frac{2z-(z_1+z_2)}{z_2-z_1}\pi\right) \neq 0$, $\forall z \in \mathbb{C}$. So the mean value theorem does not apply to $f(z)$. (This example is from [8].) □

2.6.

Proof. Suppose $f(z) = u(x,y) + iv(x,y)$. Then by Cauchy-Riemann equations, $f'(z) = 0$ in G implies all the partial derivatives of u and v with respect to x and y equal to 0 in G. Using the results for functions of real variables, we conclude u and v are constants in G. So f is a constant in G. □

2.7.

Proof. This is a direct corollary of the Cauchy-Riemann equations. □

2.8.

Proof. From the condition $au(x,y)+bv(x,y)=c$, we have

$$\begin{cases} a\frac{\partial u}{\partial x}+b\frac{\partial v}{\partial x}=0 \\ a\frac{\partial u}{\partial y}+b\frac{\partial v}{\partial y}=0. \end{cases}$$

So

$$\frac{b^2}{a^2}\frac{\partial v}{\partial x}=\left(-\frac{b}{a}\right)\frac{\partial v}{\partial x}\left(-\frac{b}{a}\right)=\frac{\partial u}{\partial x}\left(-\frac{b}{a}\right)=\frac{\partial v}{\partial y}\left(-\frac{b}{a}\right)=\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x},$$

i.e. $\left(\frac{b^2}{a^2}+1\right)\frac{\partial v}{\partial x}=0$, which implies $\frac{\partial v}{\partial x}=0$. Similarly

$$\frac{b^2}{a^2}\frac{\partial v}{\partial y}=\left(-\frac{b}{a}\right)\frac{\partial v}{\partial y}\left(-\frac{b}{a}\right)=\frac{\partial u}{\partial y}\left(-\frac{b}{a}\right)=-\frac{\partial v}{\partial x}\left(-\frac{b}{a}\right)=-\frac{\partial u}{\partial x}=-\frac{\partial v}{\partial y},$$

i.e. $\left(\frac{b^2}{a^2}+1\right)\frac{\partial v}{\partial y}=0$, which implies $\frac{\partial v}{\partial y}=0$. Combined, we conclude v is a constant in G. By Cauchy-Riemann equations, we can also conclude u is a constant in G. So $f(z)$ is a constant in G.

If a, b, and c are non-zero complex constants, the conclusion still holds. Indeed, we have shown for the real case that when $\frac{b^2}{a^2}+1\neq 0$, $f(z)$ is a constant in G. Now suppose $\frac{b^2}{a^2}+1=0$, we have two cases: $b=ia$ and $b=-ia$. In the first case, we have $af(z)=au(x,y)+iav(x,y)=au(x,y)+bv(x,y)=c$. So $f(z)=c/a$ is a constant. In the second case, we have $af^*(z)=au(x,y)-iav(x,y)=au(x,y)+bv(x,y)=c$. So $f(z)=(c/a)^*$ is a constant. □

2.9.

Proof. We write z in its polar form: $z=re^{i\theta}=r(\cos\theta+i\sin\theta)$. Then $e^z=e^{r\cos\theta}e^{i\sin\theta}$. So as z tends to ∞ with fixed argument, the argument of e^z remains the same and the modulus of e^z may tend to ∞, or 0, or remain the same, depending on the sign of $\cos\theta$.

Denote by θ the principle argument of a, then $z_n=\log|a|+(\theta+2\pi n)i$ satisfies the requirement. □

2.10.

Proof. All the equalities can be proved via the equalities for trigonometric functions and the relation between hyperbolic functions and trigonometric functions. We only prove the the two inequality.

First, we observe

$$\begin{aligned} |\sinh y| &= \frac{1}{2}|e^y-e^{-y}|=\frac{1}{2}[e^{2y}+e^{-2y}-2]^{1/2}\leq\frac{1}{2}[e^{-2y}+e^{2y}-2+4\sin^2 x]^{1/2}=|\sin(x+iy)| \\ &= \frac{1}{2}[e^{2y}+e^{-2y}+2-4\cos^2 x]^{1/2}\leq\frac{1}{2}[e^{2y}+e^{-2y}+2]^{1/2}=\cosh y. \end{aligned}$$

Similarly, we have

$$\begin{aligned} |\sinh y| &= \frac{1}{2}|e^{-y}-e^{y}|=\frac{1}{2}[e^{2y}+e^{-2y}-2]^{1/2}\leq\frac{1}{2}[e^{-2y}+e^{2y}+4\cos^2 x-2]^{1/2}=|\cos(x+iy)| \\ &= \frac{1}{2}[e^{2y}+e^{-2y}+2-4\sin^2 x]^{1/2}\leq\frac{1}{2}[e^{2y}+e^{-2y}+2]^{1/2}=\cosh y. \end{aligned}$$

□

2.11.

Proof. We first assume $|f(z)|$ is a constant in G. If this constant is 0, we have nothing to prove. So without loss of generality, we assume $|f(z)|$ is a non-zero constant in G. Suppose $f(z)=u(x,y)+iv(x,y)$, then $\frac{\partial}{\partial x}|f(z)|^2=\frac{\partial}{\partial y}|f(z)|^2=0$ gives

$$\begin{cases} 2u(x,y)\frac{\partial u(x,y)}{\partial x}+2v(x,y)\frac{\partial v(x,y)}{\partial x}=0 \\ 2u(x,y)\frac{\partial u(x,y)}{\partial y}+2v(x,y)\frac{\partial v(x,y)}{\partial y}=0,\ \forall(x,y)\in G. \end{cases}$$

Using Cauchy-Riemann equations, we have

$$\begin{cases} u\frac{\partial v}{\partial y} + v\frac{\partial v}{\partial x} = 0 \\ v\frac{\partial v}{\partial y} - u\frac{\partial v}{\partial x} = 0. \end{cases}$$

Since $\begin{vmatrix} u(x,y) & v(x,y) \\ v(x,y) & -u(x,y) \end{vmatrix} = |f(z)|^2 \neq 0$ in G, solving the above linear equations gives $\frac{\partial v(x,y)}{\partial y} = \frac{\partial v(x,y)}{\partial x} = 0$ in G. That is, $v(x,y)$ is a constant in G. Cauchy-Riemann equations imply $u(x,y)$ is a constant in G as well. So $f(z)$ is a constant in G.

We then assume $\theta = \theta(z) := \arg f(z)$ is a constant in G. Write f in polar coordinate: $f(z) = r(x,y)e^{i\theta}$. Then Cauchy Riemann equations become

$$\begin{cases} \frac{\partial}{\partial x} r(x,y)\cos\theta = \frac{\partial}{\partial y} r(x,y)\sin\theta \\ \frac{\partial}{\partial y} r(x,y)\cos\theta = -\frac{\partial}{\partial x} r(x,y)\sin\theta. \end{cases}$$

If $\sin\theta = 0$, we have $\cos\theta \neq 0$ and $\frac{\partial}{\partial x} r = \frac{\partial}{\partial y} r = 0$. If $\cos\theta = 0$, we have $\sin\theta \neq 0$ and $\frac{\partial}{\partial x} r = \frac{\partial}{\partial y} r = 0$. If $\sin\theta \neq 0$ and $\cos\theta \neq 0$, we have $\frac{\partial}{\partial x} r = \frac{\partial}{\partial y} r \tan\theta = -\frac{\partial r}{\partial x}\tan^2\theta$, which implies $\frac{\partial}{\partial x} r = 0$. Consequently, $\frac{\partial}{\partial y} r = 0$. In either of the three cases, we always have $\frac{\partial}{\partial x} r = \frac{\partial}{\partial y} r = 0$ in G. So r is a constant in G and $f(z)$ is a constant in G by the result of first half. □

2.12.

Proof. $d\xi = \frac{\partial \xi}{\partial x}dx + \frac{\partial \xi}{\partial y}dy$, $d\eta = \frac{\partial \eta}{\partial x}dx + \frac{\partial \eta}{\partial y}dy$. So by Cauchy-Riemann equations, we have

$$d\xi d\eta = \frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y}dxdy - \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}dxdy = \left(\frac{\partial \eta}{\partial y}\right)^2 dxdy + \left(\frac{\partial \xi}{\partial y}\right)^2 dxdy = |f'(z)|^2 dxdy.$$

□

2.2 Exercises at the end of chapter

1.

Proof. The basic method is to verify that Ref and Imf are differentiable as functions of real variables, and that they satisfy Cauchy-Riemann equations. □

2.

Proof. Since $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$, we have

$$\begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} := A(\theta, r) \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix}.$$

It's easy to see $A^{-1}(\theta, r) = \frac{1}{r}\begin{bmatrix} r\cos\theta & -\sin\theta \\ r\sin\theta & \cos\theta \end{bmatrix}$. Writing Cauchy-Riemann equations in matrix form, we get

$$\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} u = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} v.$$

Therefore, under the polar coordinate, the Cauchy-Riemann equations become

$$\begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \end{bmatrix} u = A(\theta, r) \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} u = A(\theta, r) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} v = A(\theta, r) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} A^{-1}(\theta, r) \begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \end{bmatrix} v = \begin{bmatrix} 0 & \frac{1}{r} \\ -r & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \end{bmatrix} v.$$

□

3.

Proof. Fix $z = re^{i\theta}$, then

$$f'(z) = \lim_{\Delta r \to 0} \frac{u(r+\Delta r,\theta) + iv(r+\Delta r,\theta) - [u(r,\theta)+iv(r,\theta)]}{\Delta r \cdot e^{i\theta}} = \frac{\partial u}{\partial r}e^{-i\theta} + i\frac{\partial v}{\partial v}e^{-i\theta} = \frac{r}{z}\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right].$$

By the result of previous problem, we have

$$\frac{r}{z}\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right] = \frac{1}{z}\left[\frac{\partial v}{\partial \theta} - i\frac{\partial u}{\partial \theta}\right].$$

□

4. We use the following result from the theory of ordinary differential equations (see, for example, Ding and Li [2], Chapter 2, Theorem 1).

Theorem 1. *Suppose function $P(x,y)$ and $Q(x,y)$ are continuous on $U = (\alpha,\beta)\times(\gamma,\delta)$, and they have continuous partial derivatives $\frac{\partial}{\partial y}P$ and $\frac{\partial}{\partial x}Q$. Then the 1-form $\omega = P(x,y)dx + Q(x,y)dy$ is exact if and only if $\frac{\partial}{\partial y}P = \frac{\partial}{\partial x}Q$ on U. Moreover, the 0-form whose differential is ω can be represented as*

$$\int_{x_0}^{x} P(\xi,y)d\xi + \int_{y_0}^{y} Q(x_0,\eta)d\eta + C,$$

where C is a constant.

(1)

Proof. $P(x,y) = \frac{\partial v(x,y)}{\partial x} = -\frac{\partial u(x,y)}{\partial y} = 2y$, $Q(x,y) = \frac{\partial v(x,y)}{\partial y} = \frac{\partial u(x,y)}{\partial x} = 2x+1$. So

$$v(x,y) = \int_0^x P(\xi,y)d\xi + \int_0^y Q(0,\eta)d\eta + C = 2xy + y + C,$$

where $C \in \mathbb{R}$ is a constant, and

$$f(z) = u(x,y) + iv(x,y) = (x^2 - y^2 + x) + i(2xy + y + C) = (x+iy)^2 + (x+iy) + iC = z^2 + z + iC.$$

□

(2)

Proof. $P(x,y) = \frac{\partial v(x,y)}{\partial x} = -\frac{\partial u(x,y)}{\partial y} = \frac{2xy}{(x^2+y^2)^2}$, $Q(x,y) = \frac{\partial v(x,y)}{\partial y} = \frac{\partial u(x,y)}{\partial x} = \frac{y^2-x^2}{(x^2+y^2)^2}$. So from some constant $C \in \mathbb{R}$,

$$\begin{aligned}
v(x,y) &= \int_1^x \frac{2\xi y}{(\xi^2+y^2)^2}d\xi + \int_0^y \frac{\eta^2-1}{(1+\eta^2)^2}d\eta + C \\
&= y\int_0^{x^2} \frac{du}{(u+y^2)^2} + \int_0^y \left[\frac{1}{1+\eta^2} - \frac{2}{(1+\eta^2)^2}\right]d\eta + C \\
&= y\left[-\frac{1}{x^2+y^2} + \frac{1}{1+y^2}\right] + \arctan y - 2\int_0^{\arctan y} \frac{d\tan\theta}{(1+\tan^2\theta)^2} + C \\
&= y\left[-\frac{1}{x^2+y^2} + \frac{1}{1+y^2}\right] + \arctan y - \int_0^{\arctan y} (\cos 2\theta + 1)d\theta + C \\
&= y\left[-\frac{1}{x^2+y^2} + \frac{1}{1+y^2}\right] - \frac{1}{2}\sin(2\arctan y) + C \\
&= y\left[-\frac{1}{x^2+y^2} + \frac{1}{1+y^2}\right] - \frac{\tan(\arctan y)}{1+\tan^2(\arctan y)} + C \\
&= -\frac{y}{x^2+y^2} + C.
\end{aligned}$$

Therefore $f(z) = u(x,y) + iv(x,y) = \frac{x-yi}{x^2+y^2} + iC = \frac{1}{z} + iC$. □

(3)

Proof. $P(x,y) = \frac{\partial v(x,y)}{\partial x} = -\frac{\partial u(x,y)}{\partial y} = -\frac{\partial(e^y \cos x)}{\partial y} = -e^y \cos x$, $Q(x,y) = \frac{\partial v(x,y)}{\partial y} = \frac{\partial u(x,y)}{\partial x} = -e^y \sin x$. So $v(x,y) = \int_0^x P(\xi,y)d\xi + \int_0^y Q(0,\eta)d\eta = -e^y \sin x + C$ where $C \in \mathbb{R}$ is a constant. So $f(z) = e^y \cos x - ie^y \sin x + iC = e^{y-ix} + iC = e^{-iz} + iC$. □

(4)

Proof. $P(x,y) = \frac{\partial v(x,y)}{\partial x} = -\frac{\partial u(x,y)}{\partial y} = -\cos x \sinh y$, $Q(x,y) = \frac{\partial v(x,y)}{\partial y} = \frac{\partial u(x,y)}{\partial x} = -\sin x \cosh y$. So $v(x,y) = -\int_0^x \cos\xi \sinh y d\xi + C = -\sin x \sinh y + C$, where $C \in \mathbb{R}$ is a constant. So $f(z) = \cos x \cosh y - i \sin x \sinh y + iC = \cos x \cos(iy) - \sin x \sin(iy) + iC = \cos(x+iy) + iC = \cos z + iC$. □

5. (1)

Proof. $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = 1 - i$. □

(2)

Proof. $f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = \cos x \cosh y - i \sin x \sinh y = \cos x \cos(iy) - \sin \sin(iy) = \cos(x+iy) = \cos z$. □

6.

Proof. We note $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = (x^2+4xy+y^2) + (x-y)(2x+4y)$ and $\frac{\partial u}{\partial y} - \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} = -(x^2+4xy+y^2) + (x-y)(4x+2y)$. Solving these two equations for $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$, we get

$$\frac{\partial u}{\partial x} = 6xy,\ \frac{\partial u}{\partial y} = 3(x^2-y^2).$$

Then it's easy to see $u(x,y) = 3x^2y - y^3 + C$ for some constant $C \in \mathbb{R}$ and consequently, $v = u - (x-y)(x^2 + 4xy + y^2) = -x^3 + 3xy^2 + C$. Therefore

$$f(z) = (3x^2y - y^3 + C) + i(-x^3 + 3xy^2 + C) = (ix-y)^3 + C(1+i) = -iz^3 + C(1+i).$$

□

7. (1)

Proof. We have $\frac{e^{iz}-e^{-iz}}{2i} = \frac{3+i}{4}$, which is equivalent to $(e^{iz})^2 - \frac{-1+3i}{2}e^{iz} - 1 = 0$. Solving this quadratic equation gives us

$$e^{iz} = \frac{\frac{-1+3i}{2} \pm \sqrt{\frac{(-1+3i)^2}{4} + 4}}{2} = \frac{(-1+3i) \pm \sqrt{8-6i}}{4} = \frac{(-1+3i) \pm (i-3)}{4}.$$

So $e^{iz} = i - 1 = e^{\frac{\ln 2}{2} + (2n\pi + \frac{3}{4}\pi)i}$ or $e^{iz} = \frac{i+1}{2} = e^{-\frac{\ln 2}{2} + (2n\pi + \frac{\pi}{4})i}$, $n \in \mathbb{Z}$. Therefore $z = \frac{3}{4}\pi + 2n\pi - \frac{i}{2}\ln 2$ or $\frac{\pi}{4} + 2n\pi + \frac{i}{2}\ln 2$, $n \in \mathbb{Z}$. □

(2)

Proof. $\cos z = 4$ is equivalent to $(e^{iz})^2 - 8e^{iz} + 1 = 0$. So $e^{iz} = 4 \pm \sqrt{15} = e^{\pm \ln(4+\sqrt{15})}$. Therefore $z = 2n\pi \pm i\ln(4+\sqrt{15})$ $(n \in \mathbb{Z})$. □

(3)

Proof. $\tan z = i$ gives $\frac{\frac{e^{iz}-e^{-iz}}{2i}}{\frac{e^{iz}+e^{-iz}}{2}} = -i\frac{e^{2iz}-1}{e^{2iz}+1} = i$. So the equation becomes $e^{2iz} - 1 = -e^{2iz} - 1$, which has no solution. □

(4)

Proof. The equation can be written as $(2\cosh z - 1)(\cosh z - 1) = 0$. So $\cosh z = 1$ or $\frac{1}{2}$. Consider the equation $\cosh z = \frac{e^z + e^{-z}}{2} = a$. We reduce it to the quadratic equation of e^z: $e^{2z} - 2ae^z + 1 = 0$. So $e^z = a \pm \sqrt{a^2 - 1}$. If $a = 1$, we get $e^z = 1$, which implies $z = 2n\pi i$ $(n \in \mathbb{Z})$. If $a = \frac{1}{2}$, we get $e^z = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$, which implies $z = (2n \pm \frac{1}{3})\pi i$ $(n \in \mathbb{Z})$. □

15.

Proof. This is Exercise 2.11. □

3 Complex Integration

1. (1)

Proof. (i) $\int_0^{2+i} \text{Re}z dz = \int_0^2 x dx + \int_2^{2+i} x dy = 2 + 2i$. (ii) $\int_0^{2+i} \text{Re}z dz = \int_0^1 2t(2dt + idt) = 2 + i$. □

(2)

Proof. (i) $\int_C \frac{dz}{\sqrt{z}} = 2\sqrt{z}|_1^{-1} = 2(e^{\frac{\pi}{2}i} - 1) = 2(i - 1)$. (ii) $\int_C \frac{dz}{\sqrt{z}} = 2\sqrt{z}|_1^{-1} = 2(e^{-\frac{\pi}{2}i} - 1) = -2(i + 1)$. □

2. (1)

Proof. By Cauchy integral formula, $\oint_{|z|=1} \frac{dz}{z} = 2\pi i \cdot 1 = 2\pi i$. □

(2)

Proof. $\oint_{|z|=1} \frac{|dz|}{z} = \int_0^{2\pi} \frac{r d\theta}{re^{i\theta}} = 0$. □

(3)

Proof. $\oint_{|z|=1} \frac{dz}{|z|} = \int_0^{2\pi} \frac{re^{i\theta} d\theta}{r} = 0$. □

(4)

Proof. $\oint_{|z|=1} \left|\frac{dz}{z}\right| = \int_0^{2\pi} d\theta = 2\pi$. □

3. (1)

Proof. Denote $\frac{1}{z^2-1} \sin \frac{\pi z}{4}$ by $f(z)$.

(i) $f(z)$ is analytic on $\{z : |z| < \frac{1}{2}\}$ and continuous on $\{z : |z| = \frac{1}{2}\}$. So by Cauchy integral theorem, $\oint_{|z|=\frac{1}{2}} f(z)dz = 0$.

(ii) $f(z)$ has one singular point $z_0 = 1$ in $\{z : |z-1| < 1\}$. So by Cauchy integral formula, $\oint_{|z-1|=1} f(z)dz = \oint_{|z-1|=1} \frac{1}{z-1} \frac{\sin \frac{\pi z}{4}}{z+1} dz = 2\pi i \frac{\sin \frac{\pi z}{4}}{z+1}|_{z=1} = \frac{\sqrt{2}}{2}\pi i$.

(iii) $f(z)$ has two singular points ± 1 in $\{z : |z| = 3\}$. So by Cauchy integral theorem for multiply connected region, $\oint_{|z|=3} f(z)dz = \oint_{|z-1|=\delta} f(z)dz + \oint_{|z+1|=\delta} f(z)dz$, where $\delta > 0$ is sufficiently small so that $\{|z-1| \le \delta\} \cup \{|z+1| \le \delta\} \subset \{|z| < 3\}$. By Cauchy integral formula, $\oint_{|z-1|=\delta} f(z)dz = 2\pi i \frac{\sin \frac{\pi z}{4}}{z+1}|_{z=1} = \frac{\sqrt{2}}{2}\pi i$, and $\oint_{|z+1|=\delta} f(z)dz = 2\pi i \frac{\sin \frac{\pi z}{4}}{z-1}|_{z=-1} = \frac{\sqrt{2}}{2}\pi i$. Combined, we conclude $\oint_{|z|=3} f(z)dz = \sqrt{2}\pi i$.

(iv) $\sqrt{2}\pi i$. The calculation is similar to that of (iii). □

(2)

Proof. Denote $\frac{1}{z^2+1}e^{iz}$ by $f(z)$.

(i) $f(z)$ has a singular point $z_0 = i$ in $\{|z-i|<1\}$. So by Cauchy integral formula, $\oint_{|z-i|=1} f(z)dz = 2\pi i \frac{e^{iz}}{z+i}|_{z=i} = \frac{\pi}{e}$.

(ii) $f(z)$ has two singular points $\pm i$ in $\{|z|<2\}$. So by Cauchy integral theorem for multiply connected region and Cauchy integral formula, for $\delta > 0$ sufficiently small,

$$\oint_{|z|=2} f(z)dz = \oint_{|z+i|=\delta} f(z)dz + \oint_{|z-i|=\delta} f(z)dz = 2\pi i\left[\frac{e^{iz}}{z-i}|_{z=-i} + \frac{e^{iz}}{z+i}|_{z=i}\right] = -2\pi\sinh 1.$$

(iii) $-2\pi\sinh 1$. The calculation is similar to that of (iii).

(iv) To have a closed curve, θ must take all the values between 0 and 4π (see figure). This closed curve forms two contours, each of which contains the two singular points $\pm i$ of $f(z)$. So by a calculation similar to that of (ii) and (iii), $\oint_{\{z=re^{i\theta}:r=3-\sin^2\frac{\theta}{4}\}} f(z)dz = 2\cdot(-2\pi\sinh 1) = -4\pi\sinh 1$.

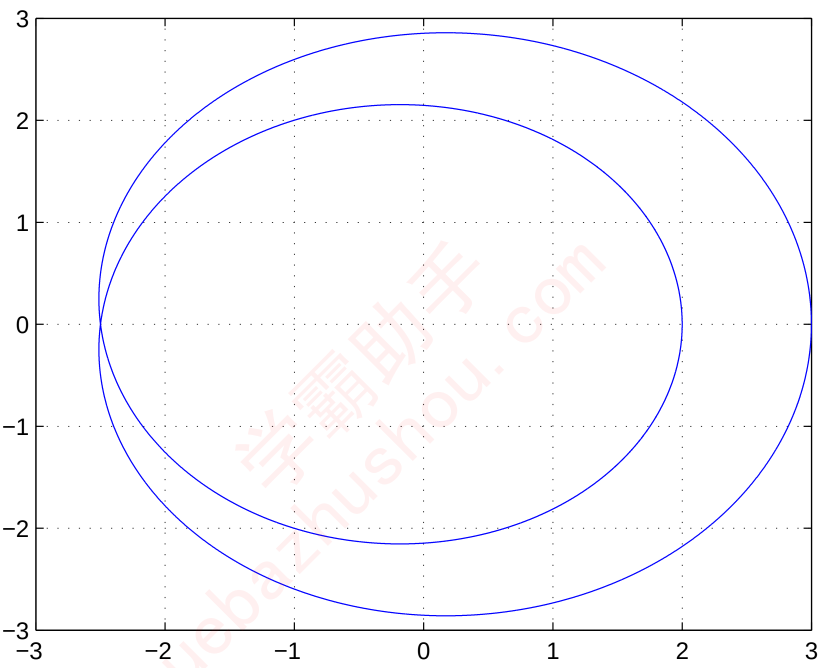

Figure 1: $r = 3-\sin^2\frac{\theta}{4}$, $\theta\in[0,4\pi]$

□

4. (1)

Proof. By Cauchy integral formula, $\oint_{|z|=2}\frac{\cos z}{z}dz = 2\pi i\cos z|_{z=0} = 2\pi i$. □

(2)

Proof. $\frac{z^2-1}{z^2+1}$ has two singular points $\pm i$ in $\{|z|<2\}$. So by Cauchy integral theorem for multiply connected region and Cauchy integral formula, for $\delta > 0$ sufficiently small,

$$\oint_{|z|=2}\frac{z^2-1}{z^2+1}dz = \oint_{|z-i|=\delta}\frac{1}{z-i}\frac{z^2-1}{z+i}dz + \oint_{|z+i|=\delta}\frac{1}{z+i}\frac{z^2-1}{z-i}dz = 2\pi i\left[\frac{z^2-1}{z+i}|_{z=i} + \frac{z^2-1}{z-i}|_{z=-i}\right] = 0.$$

□

(3)

Proof. By Cauchy integral formula, $\oint_{|z|=2} \frac{\sin e^z}{z} dz = 2\pi i \sin(e^z)|_{z=0} = 2\pi i \sin 1$. □

(4)

Proof. $\cosh z = 0$ if and only if $z = (\frac{\pi}{2} + n\pi)i$, $n \in \mathbb{Z}$. So $\{|z| < 2\}$ contains two singular points $\pm\frac{\pi}{2}i$. By Cauchy integral theorem for multiply connected region, for $\delta > 0$ sufficiently small, we have

$$\begin{aligned}
\oint_{|z|=2} \frac{e^z}{\cosh z} dz &= \oint_{|z-\frac{\pi}{2}i|=\delta} \frac{e^z}{\cosh z} dz + \oint_{|z+\frac{\pi}{2}i|=\delta} \frac{e^z}{\cosh z} dz \\
&= \oint_{|z|=\delta} \frac{e^{z+\frac{\pi}{2}i}}{\cosh(z+\frac{\pi}{2}i)} dz + \oint_{|z|=\delta} \frac{e^{z-\frac{\pi}{2}i}}{\cosh(z-\frac{\pi}{2}i)} dz \\
&= 2\oint_{|z|=\delta} \frac{e^z}{\sinh z} dz \\
&= 4\oint_{|z|=\delta} \frac{e^z}{e^z - e^{-z}} dz \\
&= 4\oint_{|z|=\delta} \frac{1}{e^{2z}-1} dz.
\end{aligned}$$

Using power series expansion of e^{2z}, we can see $\lim_{z\to 0} \frac{2z}{e^{2z}-1} = 1$ uniformly.[1] So by Lemma 3.1,

$$\lim_{\delta\to 0} \oint_{|z|=\delta} \frac{1}{e^{2z}-1} dz = 2\pi i \cdot \frac{1}{2} = \pi i.$$

So $\oint_{|z|=2} \frac{e^z}{\cosh z} dz = \lim_{\delta\to 0} 4\oint_{|z|=\delta} \frac{1}{e^{2z}-1} dz = 4\pi i$. □

5. (1)

Proof. By Cauchy integral formula,

$$\oint_{|z|=2} \frac{\sin z}{z^2} dz = 2\pi i \cdot (\sin z)'\,|_{z=0} = 2\pi i.$$

□

(2)

Proof. By Cauchy integral formula,

$$\oint_{|z|=2} \frac{|z|e^z}{z^2} dz = 2 \cdot 2\pi i \frac{1}{2\pi i} \oint_{|z|=2} \frac{e^z}{z^2} dz = 4\pi i (e^z)'|_{z=0} = 4\pi i.$$

□

(3)

Proof. By Cauchy integral formula,

$$\oint_{|z|=2} \frac{\sin z}{z^4} dz = \frac{2\pi i}{3!} \cdot \frac{3!}{2\pi i} \oint_{|z|=2} \frac{\sin z}{z^4} dz = \frac{\pi i}{3} (\sin z)^{(3)}|_{z=0} = -\frac{\pi}{3} i.$$

□

[1] We note $\left|\frac{2z}{e^{2z}-1} - 1\right| = \left|\frac{1}{1+\sum_{k=2}^{\infty} \frac{(2z)^{k-1}}{k!}} - 1\right| \le \frac{\sum_{k=2}^{\infty} \frac{|2z|^{k-1}}{k!}}{1 - \frac{|2z|^{k-1}}{k!}} \to 0$ uniformly when $z \to 0$.

(4)

Proof. By Cauchy integral formula,

$$\oint_{|z|=2} \frac{dz}{z^2(z^2+16)} = 2\pi i \left(\frac{1}{z^2+16}\right)'|_{z=0} = 0.$$

□

6. (1)

Proof. By Cauchy integral formula,

$$\oint_{|z|=1} \frac{e^z}{z^3} dz = \frac{2\pi i}{2!} \cdot \frac{2!}{2\pi i} \oint_{|z|=1} \frac{e^z}{z^3} dz = \pi i \cdot (e^z)^{(2)}|_{z=0} = \pi i.$$

□

(2)

Proof. $F(z)$ is univalent if and only if any closed curve C, $\oint_C e^z \left(\frac{1}{z} + \frac{a}{z^3}\right) dz = 0$. When the interio of C does not contain 0, this is true by Cauchy integral theorem. So we only need to consider the case where the interior of C contains 0. Without loss of generality, assume $C = \{|z| = 1\}$. Then by Cauchy integral formula

$$\int_C e^z \left(\frac{1}{z} + \frac{a}{z^3}\right) dz = \frac{2\pi i}{2!}[(z^2+a)e^z]^{(2)}|_{z=0} = a+2.$$

So when $a = -2$, $F(z)$ is univalent. □

4 Infinite Series

4.1 Exercises in the text

4.1.

Proof. (1) $a_n = \frac{1}{n \ln n}$. (2) $a_{2n} = \frac{1}{n^2+1}$, $a_{2n+1} = \frac{1}{n^2}$. (3) $a_{2n} = \frac{1}{n^3}$, $a_{2n+1} = \frac{1}{n^2}$. (4) $a_n = 0$ if n is even, $a_n = 1$ if n is odd; $b_n = 1 - a_n$. □

4.2. (1)

Proof. If $x = 0$, the series is clearly convergent. If $x \neq 0$, then by noting the series is a geometric series, we can calculate it converges to 1. To see the convergence is not uniform, note $\forall n \geq 1$

$$\left|1 - \sum_{k=1}^{n} \frac{x^2}{(1+x^2)^k}\right| = \left|1 - \frac{x^2}{1+x^2} \cdot \frac{1 - \frac{1}{(1+x^2)^n}}{1 - \frac{1}{1+x^2}}\right| = \frac{1}{(1+x^2)^n}.$$

No matter how big n is, we can always find an $x > 0$ (dependent on n), so that $\frac{1}{(1+x^2)^n} > \frac{1}{2}$. This shows the convergence is not uniform. □

(2)

Proof. We note

$$\left|\frac{(-1)^n}{n+x^2} + \frac{(-1)^{n+1}}{n+1+x^2}\right| = \frac{1}{(n+1+x^2)(n+x^2)} \leq \frac{1}{(n+1)n}.$$

So $\left|\sum_{n=N}^{\infty} \frac{(-1)^n}{n+x^2}\right| \leq \sum_{n=N}^{\infty} \frac{1}{n(n+1)}$, which implies $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+x^2}$ is uniformly convergent. But clearly, $\sum_{n=1}^{\infty} \frac{1}{n+x^2} = \infty$. □

4.3. (1)

Proof. $\min\{R_1, R_2\}$. From the special case $a_n \equiv 0$ or $b_n \equiv 0$, we can see this radius cannot be improved. □

(2)

Proof. By Cauchy's criterion, $R_1 = \underline{\lim}_{n\to\infty} |\frac{1}{a_n}|^{1/n}$ and $R_2 = \underline{\lim}_{n\to\infty} |\frac{1}{b_n}|^{1/n}$. Since for any positive sequence $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$, $\underline{\lim}_{n\to\infty} x_n \underline{\lim}_{n\to\infty} y_n \leq \underline{\lim}_{n\to\infty}(x_n y_n)$, the radius of convergence R for $\sum_{n=1}^{\infty} a_n b_n z^n$ satisfies

$$R = \underline{\lim}_{n\to\infty} \left|\frac{1}{a_n b_n}\right|^{1/n} \geq \underline{\lim}_{n\to\infty} \left|\frac{1}{a_n}\right|^{1/n} \left|\frac{1}{b_n}\right|^{1/n} = R_1 R_2.$$

The special case where $a_n = b_n \equiv 1$ shows the result $R \geq R_1 R_2$ cannot be improved. □

(3)

Proof. When $\lim_{n\to\infty} \left|\frac{1}{a_n}\right|^{1/n}$ exists, the radius of convergence is $\frac{1}{R_1}$. Otherwise, the best result we can obtain is $R = \underline{\lim}_{n\to\infty} |a_n|^{1/n} = \frac{1}{\overline{\lim}_{n\to\infty} |\frac{1}{a_n}|^{1/n}}$. □

(4)

Proof.

$$\underline{\lim}_{n\to\infty} \left|\frac{1}{\frac{b_n}{a_n}}\right|^{1/n} \geq \underline{\lim}_{n\to\infty} \left|\frac{1}{b_n}\right|^{1/n} \underline{\lim}_{n\to\infty} |a_n|^{1/n} \geq R_2 \frac{1}{\overline{\lim}_{n\to\infty} \left|\frac{1}{a_n}\right|^{1/n}}.$$

When $\lim_{n\to\infty} \left|\frac{1}{a_n}\right|^{1/n}$ exists, the right side of the above inequality becomes $\frac{R_2}{R_1}$. □

4.2 Exercises at the end of chapter

1. (1)

Proof. $\sum_{n=2}^{\infty} \left|\frac{i^n}{\ln n}\right| = \sum_{n=2}^{\infty} \frac{1}{\ln n} > \sum_{n=2}^{\infty} \frac{1}{n} = \infty$. So the series is not absolutely convergent. Meanwhile,

$$\sum_{n=2}^{\infty} \frac{i^n}{\ln n} = \sum_{k=0}^{\infty} -\left[\left(\frac{1}{\ln(4k+2)} - \frac{1}{\ln(4k+4)}\right) + i\left(\frac{1}{\ln(4k+3)} - \frac{1}{\ln(4k+5)}\right)\right].$$

By Leibnitz's criterion for the convergence of alternating series,

$$\sum_{k=0}^{\infty} -\left[\frac{1}{\ln(4k+2)} - \frac{1}{\ln(4k+4)}\right] = \sum_{m=1}^{\infty} \frac{(-1)^m}{\ln(2m)}$$

is convergent. Similarly,

$$\sum_{k=0}^{\infty} -i\left[\frac{1}{\ln(4k+3)} - \frac{1}{\ln(4k+5)}\right] = \sum_{m=1}^{\infty} i\frac{(-1)^m}{\ln(2m+1)}$$

is convergent. Therefore $\sum_{n=2}^{\infty} \frac{i^n}{\ln n}$ is convergent. □

(2)

Proof. By argument similar to that of (1), $\sum_{n=1}^{\infty} \frac{i^n}{n}$ is convergent, but not absolutely convergent. □

2.

Proof. Suppose $|z| < 1$. Then $\frac{z^{n-1}}{(1-z^n)(1-z^{n+1})} \sim z^{n-1}$. So $\sum_{n=1}^{\infty} \frac{z^{n-1}}{(1-z^n)(1-z^{n+1})}$ is absolutely convergent on $|z| < 1$. To find the sum function in this case, note for $|z| < 1$,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{z^{n-1}}{(1-z^n)(1-z^{n+1})} &= \sum_{n=1}^{\infty} z^{n-1} \frac{1}{1-z} \left(\frac{1}{1-z^n} - \frac{z}{1-z^{n+1}} \right) \\
&= \frac{1}{1-z} \left(\sum_{n=1}^{\infty} \frac{z^{n-1}}{1-z^n} - \sum_{n=1}^{\infty} \frac{z^n}{1-z^{n+1}} \right) \\
&= \frac{1}{(1-z)^2}.
\end{aligned}$$

Now suppose $|z| > 1$. Then $\frac{z^{n-1}}{(1-z^n)(1-z^{n+1})} = \frac{1}{z^{n+2}(1-z^{-n})(1-z^{-(n+1)})} \sim \frac{1}{z^{n+2}}$. So $\sum_{n=1}^{\infty} \frac{z^{n-1}}{(1-z^n)(1-z^{n+1})}$ is absolutely convergent on $|z| > 1$. To find the sum function in this case, note for $|z| > 1$,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{z^{n-1}}{(1-z^n)(1-z^{n+1})} &= \sum_{n=1}^{\infty} \frac{1}{z^{n+2}(1-z^{-n})(1-z^{-(n+1)})} \\
&= \sum_{n=1}^{\infty} \frac{1}{z^{n+2}} \frac{1}{z-1} \left(\frac{z}{1-z^{-n}} - \frac{1}{1-z^{-(n+1)}} \right) \\
&= \frac{1}{z-1} \left(\sum_{n=1}^{\infty} \frac{z^{-(n+1)}}{1-z^{-n}} - \sum_{n=1}^{\infty} \frac{z^{-(n+2)}}{1-z^{-(n+1)}} \right) \\
&= \frac{1}{z-1} \frac{z^{-2}}{1-z^{-1}} \\
&= \frac{1}{z(z-1)^2}.
\end{aligned}$$

□

3. (1)

Proof. Since for $|z| > 1$, $|z|^{n!} > |z|$ and $\sum_{n=1}^{\infty} |z| = \infty$, the series is divergent on $|z| > 1$. Since for $|z| < 1$, $|z|^{n!} < |z|^n$ and $\sum_{n=1}^{\infty} |z|^n < \infty$, the series is convergent on $|z| < 1$. □

(2)

Proof. The series converges over $\left|\frac{z}{1+z}\right| < 1$. Solving the inequality $\left|\frac{x+yi}{1+x+yi}\right| < 1$ gives $x > -\frac{1}{2}$. □

(3)

Proof. $|z^2 + 2z + 2| < 1$. □

(4)

Proof. Suppose $z = x + yi$. Then

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} = \frac{1}{2i}(e^{-y+ix} - e^{y-ix}) = \frac{1}{2i}\left[e^{ix}(e^{-y} - e^y) + e^y(e^{ix} - e^{-ix})\right] = \frac{1}{2i}e^{ix}(e^{-y} - e^y) + e^y \sin x.$$

As $z \to 0$, $e^y \sin x \sim x$ and $\frac{1}{2i}e^{ix}(e^{-y} - e^y) \sim \frac{1}{2i}(-2y) = yi$. So $\sum_{n=1}^{\infty} 2^n \left[\frac{1}{2i} e^{i\frac{x}{3^n}} (e^{-\frac{y}{3^n}} - e^{\frac{y}{3^n}})\right]$ is convergent if and only if $\sum_{n=1}^{\infty} 2^n \frac{y}{3^n} i$ is convergent and $\sum_{n=1}^{\infty} 2^n e^{\frac{y}{3^n}} \sin \frac{x}{3^n}$ is convergent if and only if $\sum_{n=1}^{\infty} 2^n \frac{x}{3^n}$ is convergent. Since $\sum_{n=1}^{\infty} 2^n \sin \frac{z}{3^n} = \sum_{n=1}^{\infty} 2^n \left[\frac{1}{2i} e^{i\frac{y}{3^n}} (e^{-\frac{y}{3^n}} - e^{\frac{y}{3^n}}) + e^{\frac{y}{3^n}} \sin \frac{x}{3^n}\right]$, and since for arbitrary $x, y \in \mathbb{R}$, $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n yi$ and $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n x$ both converge, we conclude $\forall z \in \mathbb{C}$, $\sum_{n=1}^{\infty} 2^n \sin \frac{z}{3^n}$ is convergent. □

4.

Proof. At $z = 1$, the sum is equal to $\sum_{n=0}^{\infty}\left(\frac{1}{n+1} - \frac{2}{2n+3}\right) = \sum_{n=0}^{\infty}\left(\frac{2}{2n+2} - \frac{2}{2n+3}\right) = 2(-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \cdots + 1) = 2(1 - \ln 2)$. For $|z| < 1$, $\sum_{n=0}^{\infty}\frac{z^{n+1}}{n+1} = \sum_{n=0}^{\infty}\int_0^z w^n dw = \int_0^z(\sum_{n=0}^{\infty} w^n)dw = \int_0^z \frac{dw}{1-w}$. Similarly, $\sum_{n=0}^{\infty}\frac{2z^{2n+3}}{2n+3} = \sum_{n=0}^{\infty}\int_0^z 2w^{2n+2}dw = \int_0^z 2w^2\frac{dw}{1-w^2} = -2z + \int_0^z\frac{dw}{1-w} - \int_0^z\frac{dw}{1+w}$. So $\sum_{n=0}^{\infty}\left(\frac{z^{n+1}}{n+1} - \frac{2z^{2n+3}}{2n+3}\right) = 2z - \ln(1+z)$ for $|z| < 1$. In summary,

$$S(z) = \sum_{n=0}^{\infty}\left(\frac{z^{n+1}}{n+1} - \frac{2z^{2n+3}}{2n+3}\right) = \begin{cases} 2z - \ln(1+z), & |z| < 1 \\ 2 - 2\ln 2, & z = 1. \end{cases}$$

□

5.

Proof.

$$\ln(1-z) = \int_0^z \frac{-dw}{1-w} = -\int_0^z \sum_{n=0}^{\infty} w^n dw = -\sum_{n=0}^{\infty}\frac{z^{n+1}}{n+1}.$$

So for any $r \in (-1, 1)$, by letting $z = re^{i\theta}$, we get

$$\ln(1 + re^{i\theta}) = -\sum_{n=1}^{\infty}\frac{(-re^{i\theta})^n}{n} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}r^n e^{in\theta} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(r^n\cos n\theta + ir^n\sin n\theta).$$

Suppose $\ln(1 + re^{i\theta}) = x + yi$. Then $e^x\cos y = 1 + r\cos\theta$ and $e^x\sin y = r\sin\theta$. Solving for x and y gives $x = \frac{1}{2}\ln(1 + 2r\cos\theta + r^2)$, $y = \arctan\frac{r\sin\theta}{1+r\cos\theta}$. Matching the real part and imaginary part in the equality

$$\frac{1}{2}\ln(1 + 2r\cos\theta + r^2) + i\arctan\frac{r\sin\theta}{1 + r\cos\theta} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(r^n\cos n\theta + ir^n\sin n\theta)$$

gives the desired equalities. □

6. (1)

Proof. In the first equality of Problem 5, replace θ with $\theta + \pi$ and let $r \to 1$, we get

$$\cos\theta + \frac{\cos 2\theta}{2} + \frac{\cos 3\theta}{3} + \cdots = -\frac{1}{2}\ln(2 - 2\cos\theta) = -\ln(2\sin\frac{\theta}{2}).$$

To justify the process of taking limit, according to Abel's second theorem, it suffices to show the series $\sum_{n=1}^{\infty}\frac{\cos n\theta}{n}$ is convergent. Since $(\frac{1}{n})_{n\ge 1}$ is monotone decreasing to 0, by Dirichlet's criterion, it suffices to show $\sum_{k=1}^{n}\cos k\theta$ is bounded for any n. This can be seen by noting $\frac{1-e^{i(n+1)\theta}}{1-e^{i\theta}} = \sum_{k=0}^{n}e^{ik\theta}$. After writing both sides in terms of real and imaginary parts, we have by comparison

$$\left|\sum_{k=1}^{n}\cos k\theta\right| = \left|\frac{\cos(n+2)\theta - \cos(n+1)\theta}{2(1-\cos\theta)} - \frac{1}{2}\right| \le \frac{1}{1-\cos\theta} + \frac{1}{2}.$$

Similarly, in the second equality of Problem 5, replace θ with $\theta + \pi$ and let $r \to 1$, we get

$$\sum_{n=1}^{\infty}\frac{\sin n\theta}{n} = \arctan\frac{\sin\theta}{1-\cos\theta} = \arctan\cot\frac{\theta}{2} = \frac{1}{2}(\pi - \theta).$$

□

(2)

Proof. Let $r \to 1$ in the first equality of Problem 5, we have $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos n\theta}{n} = \frac{1}{2}\ln(2+2\cos\theta)$. In (i), we have shown $\sum_{n=1}^{\infty}\frac{\cos n\theta}{n} = -\ln(2\sin\frac{\theta}{2})$. Add up these two equalities and divide both sides by 2, we get (note $1+\cos\theta = 2\cos^2\frac{\theta}{2}$)

$$\cos\theta + \frac{\cos 3\theta}{3} + \frac{\cos 5\theta}{5} + \frac{\cos 7\theta}{7} + \cdots = \frac{1}{2}\ln\cot\frac{\theta}{2}.$$

Similar argument gives (note $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$)

$$\begin{aligned} & \sin\theta + \frac{\sin 3\theta}{3} + \frac{\sin 5\theta}{5} + \frac{\sin 7\theta}{7} + \cdots \\ = & \frac{1}{2}\left(\arctan\frac{\sin\theta}{1+\cos\theta} + \frac{1}{2}(\pi-\theta)\right) \\ = & \frac{1}{2}\left[\arctan\left(\tan\frac{\theta}{2}\right) + \frac{1}{2}(\pi-\theta)\right] \\ = & \frac{\pi}{4}. \end{aligned}$$

□

(3)

Proof. Integrating from $\frac{\pi}{2}$ to θ on both sides of

$$\sin\theta + \frac{\sin 3\theta}{3} + \frac{\sin 5\theta}{5} + \frac{\sin 7\theta}{7} + \cdots = \frac{\pi}{4},$$

we get

$$\cos\theta + \frac{\cos 3\theta}{3^2} + \frac{\cos 5\theta}{5^2} + \frac{\cos 7\theta}{7^2}\cdots = \frac{\pi}{4}\left(\frac{\pi}{2}-\theta\right).$$

Replace θ with $\theta+\frac{\pi}{2}$, we get

$$-\sin\theta + \frac{\sin 3\theta}{3^2} - \frac{\sin 5\theta}{5^2} + \frac{\sin 7\theta}{7^2}\cdots = \frac{\pi}{4}(-\theta).$$

Therefore

$$\sin\theta - \frac{\sin 3\theta}{3^2} + \frac{\sin 5\theta}{5^2} - \frac{\sin 7\theta}{7^2} + \cdots = \frac{\pi}{4}\theta.$$

□

(4)

Proof. From part (2), we know

$$\cos\theta + \frac{\cos 3\theta}{3} + \frac{\cos 5\theta}{5} + \frac{\cos 7\theta}{7} + \cdots = \frac{1}{2}\ln\cot\frac{\theta}{2}.$$

Form part (3), we obtain by differentiation

$$\cos\theta - \frac{\cos 3\theta}{3} + \frac{\cos 5\theta}{5} - \frac{\cos 7\theta}{7} + \cdots = \frac{\pi}{4}.$$

□

7.

Proof. (1) $R = \underline{\lim}_{n\to\infty} |n^n|^{1/n} = +\infty$.

(2) $R = \infty$.

(3) $R = \lim_{n\to\infty} \frac{n!/n^n}{(n+1)!/(n+1)^{n+1}} = \lim_{n\to\infty} \frac{1}{n+1}(1+\frac{1}{n})^n \cdot (n+1) = e$.

(4) $R = \infty$.

(5) $R = \lim_{n\to\infty}(n^{-\ln n})^{1/n} = \lim_{n\to\infty}(e^{\ln n \cdot \ln n^{-1}})^{1/n} = \lim_{n\to\infty} e^{-\frac{(\ln n)^2}{n}} = 1$.

(6) $R = 2$.

(7) $R = \lim_{n\to\infty} \frac{\ln n^n/n!}{\ln(n+1)^{n+1}/(n+1)!} = \lim_{n\to\infty}(n+1)\frac{n\ln n}{(n+1)\ln(n+1)} = \infty$.

(8) $R = \lim_{n\to\infty} |(1-\frac{1}{n})^{-n}|^{1/n} = \lim_{n\to\infty}(1-\frac{1}{n})^{-1} = 1$. □

5 Local Expansion of Analytic Functions

5.1 Exercises in the text

5.1.

Proof. If $g(z) \equiv 0$, the problem is ill-posed; if $f(z) \equiv 0$, there is nothing to prove. So without loss of generality, we can assume z_0 is a zero of $f(z)$ of order n and a zero of $g(z)$ of order m. Then $f(z)$ can be written as $f(z) = (z-z_0)^n\phi(z)$ where $\phi(z)$ is analytic at z_0 and is non-zero in a neighborhood of z_0. Similarly, $g(z)$ can be written as $g(z) = (z-z_0)^m\psi(z)$ where $\psi(z)$ is analytic at z_0 and is non-zero in a neighborhood of z_0. Then we have

$$\frac{f(z)}{g(z)} = \begin{cases} (z-z_0)^{n-m}\cdot\frac{\phi(z)}{\psi(z)} & \text{if } n > m \\ \frac{\phi(z)}{\psi(z)} & \text{if } n = m \\ \frac{1}{(z-z_0)^{m-n}}\cdot\frac{\phi(z)}{\psi(z)} & \text{if } n < m, \end{cases}$$

which implies

$$\lim_{z\to z_0}\frac{f(z)}{g(z)} = \begin{cases} 0 & \text{if } n > m \\ \frac{\phi(z_0)}{\psi(z_0)} & \text{if } n = m \\ \infty & \text{if } n < m. \end{cases}$$

Similarly, we have

$$\frac{f'(z)}{g'(z)} = \frac{n(z-z_0)^{n-1}\phi(z)+(z-z_0)^n\phi'(z)}{m(z-z_0)^{m-1}\psi(z)+(z-z_0)^m\psi'(z)} = \begin{cases} \frac{n(z-z_0)^{n-m}\phi(z)+(z-z_0)^{n-m+1}\phi'(z)}{m\psi(z)+(z-z_0)\psi'(z)} & \text{if } n > m \\ \frac{n\phi(z)+(z-z_0)\phi'(z)}{n\psi(z)+(z-z_0)\psi'(z)} & \text{if } n = m \\ \frac{n\phi(z)+(z-z_0)\phi'(z)}{m(z-z_0)^{m-n}\psi(z)+(z-z_0)^{m-n+1}\psi'(z)} & \text{if } n < m, \end{cases}$$

which implies

$$\lim_{z\to z_0}\frac{f'(z)}{g'(z)} = \begin{cases} 0 & \text{if } n > m \\ \frac{\phi(z_0)}{\psi(z_0)} & \text{if } n = m \\ \infty & \text{if } n < m. \end{cases}$$

Therefore we must have

$$\lim_{z\to z_0}\frac{f(z)}{g(z)} = \lim_{z\to z_0}\frac{f'(z)}{g'(z)}.$$

□

5.2.

Proof. The solution is similar to that of problem 5.1 once we write $f(z)$ and $g(z)$ as $\frac{\phi(z)}{(z-z_0)^n}$ and $\frac{\psi(z)}{(z-z_0)^m}$, respectively. □

5.3.

Proof. $z_n = \frac{1}{(2n+1)\pi i}$, $z_n = \frac{1}{2(n+1)\pi i}$, $z_n = \frac{1}{(2n+\frac{1}{2})\pi i}$, and $z_n = \frac{1}{(2n-\frac{1}{2})\pi i}$. □

5.4.

Proof. By considering the singularity of 0 for $f(1/z)$, we can conclude the following results: if ∞ is a removable singularity of $f(z)$, then $f(z)$ has the form of $\sum_{n=0}^{\infty} \frac{a_n}{z^n}$ in a neighborhood of ∞; if ∞ is a pole of $f(z)$, then $f(z)$ has the form of $\sum_{n=0}^{\infty} \frac{a_n}{z^n} + \sum_{k=1}^{m} b_k z^k$ for some positive integer m in a neighborhood of ∞ and $b_m \neq 0$; finally, if ∞ is an essential singularity of $f(z)$, then $f(z)$ has the form of $\sum_{n=0}^{\infty} \frac{a_n}{z^n} + \sum_{k=1}^{\infty} b_k z^k$ where infinitely many b_k's are non-zero. □

5.5.

Proof. In formula (5.48), we already obtained

$$\operatorname{sech}\frac{z}{2} = \sum_{n=0}^{\infty} \frac{E_n}{n!}\left(\frac{z}{2}\right)^n, \ |z| < \pi.$$

Replace $\frac{z}{2}$ by iw with $|w| < \frac{\pi}{2}$. □

5.2 Exercises at the end of chapter

1. (1)

Proof. $1 - z^2 = 1 - [(z-1)+1]^2 = -2(z-1) - (z-1)^2$. Radius of convergence $R = \infty$. □

(2)

Proof. $\sin z = (-1)^n \sin(z - n\pi) = (-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}(z-n\pi)^{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^{n+k}}{(2k+1)!}(z-n\pi)^{2k+1}$. Radius of convergence $R = \infty$. □

(3)

Proof. Solve the equation $1 + z + z^2$ to get two roots: $z_1 = e^{\frac{2}{3}\pi i} = \frac{-1+\sqrt{3}i}{2}$ and $z_2 = e^{\frac{4}{3}\pi i} = \frac{-1-\sqrt{3}i}{2}$. Then

$$\begin{aligned}
\frac{1}{1+z+z^2} &= \frac{1}{z_1 - z_2}\left(\frac{1}{z-z_1} - \frac{1}{z-z_2}\right) \\
&= \frac{-1}{\sqrt{3}i}\left(\frac{1}{z_1}\frac{1}{1-\frac{z}{z_1}} - \frac{1}{z_2}\frac{1}{1-\frac{z}{z_2}}\right) \\
&= \frac{i}{\sqrt{3}}\left[\frac{1}{z_1}\sum_{n=0}^{\infty}\left(\frac{z}{z_1}\right)^n - \frac{1}{z_2}\sum_{n=0}^{\infty}\left(\frac{z}{z_2}\right)^n\right] \\
&= \frac{i}{\sqrt{3}}\sum_{n=0}^{\infty} z^n \left(e^{-\frac{2}{3}(n+1)\pi i} - e^{-\frac{4}{3}(n+1)\pi i}\right) \\
&= -\frac{2}{\sqrt{3}}\sum_{n=0}^{\infty} z^n \frac{e^{-\frac{2}{3}(n+1)\pi i} - e^{\frac{2}{3}(n+1)\pi i}}{2i} \\
&= \sum_{n=0}^{\infty} \frac{\sin\frac{2}{3}(n+1)\pi}{\sin\frac{2}{3}\pi} z^n.
\end{aligned}$$

The radius of convergence $R = 1$. □

(4)

Proof.

$$\frac{\sin z}{1-z} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k+1} \cdot \sum_{m=0}^{\infty} z^m = \sum_{n=1}^{\infty} \left[\sum_{2k+1+m=n} \frac{(-1)^k}{(2k+1)!} \right] z^n = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{[(n-1)/2]} \frac{(-1)^k}{(2k+1)!} \right] z^n.$$

The radius of convergence $R = 1$. □

(5)

Proof. It's clear that 1 is a singularity. So the radius of convergence $R = 1$ and the function is analytic inside the unit disc $D(0,1)$. Suppose $e^{\frac{1}{1-z}} = \sum_{k=0}^{\infty} a_k z^k$. Differentiate both sides and we get $\frac{e^{\frac{1}{1-z}}}{(1-z)^2} = \sum_{k=1}^{\infty} k a_k z^{k-1}$. Therefore

$$\sum_{k=0}^{\infty} a_k z^k = e^{\frac{1}{1-z}} = (1 - 2z + z^2) \sum_{k=1}^{\infty} k a_k z^{k-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n - 2 \sum_{n=1}^{\infty} n a_n z^n + \sum_{n=2}^{\infty} (n-1) a_{n-1} z^n.$$

So $a_0 = a_1$, $a_1 = 2a_2 - 2a_1$ and $a_{n+1} = \frac{2n+1}{n+1} a_n - \frac{n-1}{n+1} a_{n-1}$ for $n \geq 2$. By recursion, we get $a_0 = e$, $a_1 = e$, $a_2 = \frac{3}{2}e$, $a_3 = \frac{13}{6}e$, and $a_4 = \frac{73}{24}e$. So $e^{\frac{1}{1-z}} = e\left(1 + z + \frac{3}{2}z^2 + \frac{13}{6}z^3 + \frac{73}{24}e^4 + O(z^5)\right)$. □

2. (1)

Proof. Since $z = 0$ is a singularity for $\ln z$, the radius of convergence $R = 1$. Using the power series of $\ln(1-z)$, we have

$$\ln z = \ln[i(1 - i(z-i))] = \ln i - \sum_{n=0}^{\infty} \frac{[i(z-i)]^n}{n} = \frac{\pi}{2} i - \sum_{n=0}^{\infty} \frac{i^n}{n} (z-i)^n.$$

□

(2)

Proof. The solution is similar to that of problme (1), only that $\ln i = -\frac{3}{2}\pi$. □

(3)

Proof. We note $\arctan z$ is an od function, so its series expansion must have the form $\sum_{n=0}^{\infty} a_{2n+1} z^{2n+1}$. Differentiate both sides and we get $\frac{1}{1+z^2} = \sum_{n=0}^{\infty} a_{2n+1}(2n+1) z^{2n}$. So

$$1 = (1+z^2) \sum_{n=0}^{\infty} a_{2n+1}(2n+1) z^{2n} = \sum_{n=0}^{\infty} a_{2n+1}(2n+1) z^{2n} + \sum_{n=1}^{\infty} a_{2n-1}(2n-1) z^{2n}.$$

Therefore $a_1 = 1$ and $a_{2n+1} = -\frac{2n-1}{2n+1} a_{2n-1}$ for $n \geq 1$. This implies $a_{2n+1} = \frac{(-1)^n}{2n+1}$. Since we have used $\frac{1}{1+z^2}$ which has $\pm i$ as poles, the radius of convergence $R = 1$. □

(4)

Proof.

$$\begin{aligned} \ln \frac{1+z}{1-z} &= \ln(-1) + \ln\left(1 + \frac{1}{z}\right) - \ln\left(1 - \frac{1}{z}\right) \\ &= (2k+1)\pi i + \sum_{n=1}^{\infty} \left[(-1)^{n+1} + 1\right] \frac{z^{-n}}{n} \\ &= (2k+1)\pi i + \sum_{n=0}^{\infty} \frac{2}{2n+1} z^{-(n+1)}. \end{aligned}$$

We used the assumption $\frac{1}{|z|} < 1$ in the above derivation, so the domain of convergence is $|z| > 1$. □

3. (1)

Proof. Let $f(z) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{2n+1}$. Then

$$f'(z) = \sum_{n=0}^{\infty} z^{2n} = \frac{1}{1-z^2} = \frac{1}{2}\left(\frac{1}{1-z} + \frac{1}{1+z}\right).$$

Therefore $f(z) = \frac{1}{2}\ln\frac{1+z}{1-z}$ with $f(0) = 0$. □

(2)

Proof. Inspired by definition of trigonometric functions via exponential function, we have

$$\sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} = \frac{1}{2}\left[\sum_{n=0}^{\infty} \frac{z^n}{n!} + \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{n!}\right] = \frac{1}{2}(e^z + e^{-z}) = \cosh z.$$

□

4. (1)

Proof. Assume $\frac{1}{z^2(z-1)} = \sum_{n=-\infty}^{\infty} a_n (z-1)^n$. Then by Theorem 5.4,

$$a_n = \frac{1}{2\pi i}\int_{|z-1|=\rho} \frac{1}{z^2(z-1)} \cdot \frac{dz}{(z-1)^{n+1}} = \frac{1}{2\pi i}\int_{|z-1|=\rho} \frac{1}{z^2} \cdot \frac{dz}{(z-1)^{n+2}}.$$

If $n \le -2$, by Cauchy's theorem, $a_n = 0$. If $n = -1$, by Cauchy's integral formula, $a_n = \frac{1}{z^2}|_{z=1} = 1$. If $n \ge 0$, by Cauchy's integral formula

$$a_n = \frac{1}{(n+1)!}\frac{d^{n+1}}{dz^{n+1}}(z^{-2})\Big|_{z=1} = \frac{1}{(n+1)!}(-2)(-3)\cdots[-(n+2)]z^{-(n+3)}\Big|_{z=1} = (n+2)(-1)^{n+1}.$$

So $\frac{1}{z^2(z-1)} = \sum_{n=0}^{\infty}(n+2)(-1)^{n+1}(z-1)^n + (z-1)^{-1} = \sum_{n=-1}^{\infty}(-1)^{n+1}(n+2)(z-1)^n$. □

(2)

Proof.

$$\frac{1}{z^2(z-1)} = \frac{1}{z^3} \cdot \frac{1}{1-\frac{1}{z}} = \frac{1}{z^3}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n = \sum_{n=3}^{\infty} z^{-n}.$$

□

(3)

Proof.

$$\begin{aligned}\frac{1}{z^2-3z+2} &= \frac{1}{z-2} - \frac{1}{z-1} = -\frac{1}{2(1-\frac{z}{2})} - \frac{1}{z(1-\frac{1}{z})} \\ &= -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n - \sum_{n=0}^{\infty}\frac{1}{z^{n+1}} = -\sum_{n=-1}^{-\infty} z^n - \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}.\end{aligned}$$

□

(4)

Proof.

$$\frac{1}{z^2-3z+2}=\frac{1}{z-2}-\frac{1}{z-1}=\frac{1}{z}\left(\frac{1}{1-\frac{2}{z}}-\frac{1}{1-\frac{1}{z}}\right)=\sum_{n=1}^{\infty}(2^{n-1}-1)z^{-n}.$$

□

(5)

Proof.

$$\begin{aligned}\frac{(z-1)(z-2)}{(z-3)(z-4)} &= (z^2-3z+2)\left(-\frac{1}{4(1-\frac{z}{4})}-\frac{1}{z(1-\frac{3}{z})}\right)\\ &= (z^2-3z+2)\left[-\frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{z}{4}\right)^n-\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{3}{z}\right)^n\right]\\ &= 1-\frac{3}{2}\sum_{n=0}^{\infty}\frac{z^n}{2^{2n}}-2\sum_{n=-1}^{-\infty}\frac{z^n}{3^{n+1}}.\end{aligned}$$

□

(6)

Proof.

$$\begin{aligned}\frac{(z-1)(z-2)}{(z-3)(z-4)} &= (z^2-3z+2)\left(\frac{1}{z-4}-\frac{1}{z-3}\right)\\ &= \left(z-3+\frac{2}{z}\right)\left[\sum_{n=0}^{\infty}\left(\frac{4}{z}\right)^n-\sum_{n=0}^{\infty}\left(\frac{3}{z}\right)^n\right]\\ &= 1+\sum_{n=1}^{\infty}(3\cdot 2^{2n-1}-2\cdot 3^{n-1})z^{-n}.\end{aligned}$$

□

6. (1)

Proof. $\frac{1}{z^2+a^2}=\frac{1}{2ai}\left[\frac{1}{z-ai}-\frac{1}{z+ai}\right]$. So $\pm ai$ are poles of order 1. Since $\lim_{z\to\infty}\frac{1}{z^2+a^2}=0$, ∞ is a removable singularity. □

(2)

Proof.

$$\frac{\cos az}{z^2}=\frac{\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}z^{2n}}{z^2}=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}z^{2n-2}.$$

So 0 is a pole of order 2 and ∞ is an essential singularity. □

(3)

Proof.

$$\frac{\cos az-\cos bz}{z^2}=\frac{1}{z^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left[a^{2n}-b^{2n}\right]z^{2n}=\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)!}\left[a^{2n}-b^{2n}\right]z^{2n-2}.$$

So 0 is a removable singularity and ∞ is an essential singularity. □

(4)

Proof.

$$\frac{\sin z}{z^2}-\frac{1}{z}=\frac{\sin z-z}{z^2}=\frac{\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n+1}}{z^2}=\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n-1}.$$

So 0 is a removable singularity and ∞ is an essential singularity. □

(5)

Proof.

$$\cos\frac{1}{\sqrt{z}}=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(\sqrt{z})^{2n}=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}z^n.$$

So ∞ is an essential singularity. □

(6)

Proof.

$$\frac{\sqrt{z}}{\sin\sqrt{z}}=\frac{\sqrt{z}}{\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^n\cdot\sqrt{z}}=\frac{1}{\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^n}.$$

So 0 is a removable singularity. Meanwhile, $(n\pi)^2$ $(n\in\mathbb{N})$ are poles of order 1 and ∞ is a non-isolated singularity. □

(7)

Proof. If we stipulate $\ln z|_{z=1}\neq 0$, then 1 is a pole of order 1. If we stipulate $\ln z|_{z=1}=0$, then by l'Hospitale rule for analytic functions (Exercise problem 5.1 and 5.2 in the text), we have

$$\lim_{z\to 1}\frac{z-1}{\ln z-\ln 1}=\lim_{z\to 1}z=1.$$

So $\frac{1}{(z-1)\ln z}=\frac{1}{(z-1)^2}\cdot\frac{z-1}{\ln z}$ has 1 as a pole of order 2. ∞ is a removable singularity. □

(8)

Proof.

$$\int_0^z\frac{\sinh\sqrt{\zeta}}{\sqrt{\zeta}}d\zeta=\int_0^z\sum_{k=1}^{\infty}\frac{\zeta^{k-1}}{(2k-1)!}d\zeta=\sum_{k=1}^{\infty}\frac{z^k}{k(2k-1)!}.$$

So the function is an entire function with ∞ an essential singularity. □

7.

Proof. z^2 has ∞ as a pole of order of 2. $\frac{1}{z}$ has ∞ as a removable singularity. $\frac{\cos z}{z}$ has ∞ as an essential singularity (see problem 6(2)). $\frac{z}{\cos z}$ has ∞ as a non-isolated singularity since $n\pi+\frac{\pi}{2}$ $(n\in\mathbb{Z})$ are poles. $\frac{z^2+1}{e^z}$ has ∞ as an essential singularity since $\lim_{z\to\infty}\frac{z^2+1}{e^z}$ does not exist: $\lim_{z\to\infty,z\in\mathbb{R}}\frac{z^2+1}{e^z}=0\neq\infty=\lim_{z\to\infty,z\in\mathbb{R}i}\frac{z^2+1}{e^z}$. $\exp\{-\frac{1}{z^2}\}$ has ∞ as a removable singularity since $\lim_{z\to\infty}\exp\{-\frac{1}{z^2}\}=e^0=1$. $\frac{1}{\cosh\sqrt{z}}$ has $-(n\pi+\frac{\pi}{2})^2$ as poles $(n\in\mathbb{Z})$, so it has ∞ as a non-isolated singularity. The function $\sqrt{\left(\frac{1}{z}-1\right)\left(\frac{1}{z}-2\right)}=\frac{\sqrt{(1-z)(1-2z)}}{z}$ has 0 as a pole of order 1, so the function $\sqrt{(z-1)(z-2)}$ has ∞ as a pole of order 1. □

8.

Proof. The series $\sum_{n=0}^{\infty}(\alpha z)^n$ is convergent in $U_1=\{z:|z|<\frac{1}{|\alpha|}\}$. The series $\frac{1}{1-z}\sum_{n=0}^{\infty}(-1)^n\frac{[(1-\alpha)z]^n}{(1-z)^n}$ is convergent in $U_2=\{z:|1-\alpha||z|<|1-z|\}$. On $U_3=U_1\cap U_2$, both of the series represent the analytic function $\frac{1}{1-\alpha z}$. So the functions represented by these two series are analytic continuation of each other. □

9.

Proof. Denote by $f(z)$ the analytic function represented by the series in $|z|<1$ and by $g(z)$ the analytic function represented by the series in $|z|>1$. Since the roots of $z^n=1$ ($n\in\mathbb{N}$) are singularities of the series and consist of a dense subset of $\partial D(0,1)$, $f(z)$ and $g(z)$ cannot be analytic continuation of each other. Of course, we need to verify that the series indeed converges to analytic functions both in $|z|<1$ and in $|z|>1$. When $|z|\le\rho<1$, we note

$$\left|\frac{1}{1-z^{n+1}}-\frac{1}{1-z^n}\right|=\left|\frac{z^n(z-1)}{(1-z^{n+1})(1-z^n)}\right|\le\frac{\rho^n}{(1-\rho^n)(1-\rho^{n+1})}\le\frac{\rho^n}{(1-\rho)(1-\rho)}.$$

By Weierstrass criterion, the series is uniformly convergent to an analytic function on $|z|\le\rho$. So $f(z)$ is well-defined in $|z|<1$. When $|z|\ge R>1$, we note

$$\begin{aligned}\left|\frac{1}{1-z^{n+1}}-\frac{1}{1-z^n}\right| &= \left|\frac{1}{z^{n+1}(\frac{1}{z^{n+1}}-1)}-\frac{1}{z^n(\frac{1}{z^n}-1)}\right|\\ &\le \frac{1}{R^{n+1}}\frac{1}{1-\frac{1}{R^{n+1}}}+\frac{1}{R^n}\frac{1}{1-\frac{1}{R^n}}\\ &\le \frac{1}{R^{n+1}}\frac{1}{1-\frac{1}{R}}+\frac{1}{R^n}\frac{1}{1-\frac{1}{R}}.\end{aligned}$$

By Weierstrass criterion, the series is uniformly convergent to an analytic function on $|z|\ge R$. So $g(z)$ is well-defined in $|z|>1$. □

10.

Proof. Clearly the series is convergent in $D(0,1)$ and is divergent at $z=1$. So its radius of convergent $R=1$. Then $f(z)$ is analytic in $D(0,1)$ and $z=1$ is a singularity of $f(z)$. We note $f(z)=z+\sum_{n=1}^\infty z^{2^n}=z+\sum_{n=1}^\infty z^{2\cdot 2^{n-1}}=z+\sum_{n=1}^\infty (z^2)^{2^{n-1}}=z+f(z^2)$. Therefore, we have

$$f(z)=z+f(z^2)=z+z^2+f(z^4)=z+z^2+z^4+f(z^8)=\cdots.$$

So the roots of equations $z^2=1$, $z^4=1$, $z^8=1,\cdots$, $z^{2^n}=1$, $\cdots$, etc. are all singularities of f. These roots form a dense subset of $\partial D(0,1)$, so $f(z)$ can not be analytically continued to the outside of its circle of convergence $D(0,1)$. □

6 Power Series Solution of Second Order Linear ODE

6.1 Exercises in the text

6.1.

Proof. It suffices to note we have the following linear equations

$$\begin{cases}p(z)w_1'(z)+q(z)w_1(z)=-w_1''(z)\\ p(z)w_2'(z)+q(z)w_2(z)=-w_2''(z).\end{cases}$$

Then we can apply results in linear algebra (i.e. Cramer's rule). □

6.2 Exercises at the end of chapter

1. (1)

Proof. Let $w(z)=c_0w_1(z)+c_1w_2(z)$. Then $w'(z)=c_0+c_1e^z$ and $w''(z)=c_1e^z$. Then $w(z)-zw'(z)=c_1e^z-c_1ze^z=(1-z)w''$. So the differential equation satisfied by $w(z)$ is

$$(z-1)w''-zw'+w=0.$$

□

(2)

Proof. Let $w(z) = c_0 w_1(z) + c_1 w_2(z)$. Then

$$w' = -\frac{c_0}{z^2} w_1 + \frac{2c_1}{z^2} w_2.$$

So $z^2 w' = -c_0 e^{\frac{1}{z}} + 2c_1 e^{-\frac{2}{z}}$ (*). Differentiating both sides and multiply both sides by z^2, we get $2z^3 w' + z^4 w'' = c_0 e^{\frac{1}{z}} + 4c_1 e^{-\frac{2}{z}}$ (**). Combining (*) and (**) gives us $6c_1 e^{-\frac{2}{z}} = (z^2 + 2z^3)w' + z^4 w''$ and $3c_0 e^{\frac{1}{z}} = (2z^3 - 2z^2)w' + z^4 w''$. So

$$w = \frac{1}{3}[(2z^3 - 2z^2)w' + z^4 w''] + \frac{1}{6}[(z^2 + 2z^3)w' + z^4 w''] = \left(z^3 - \frac{z^2}{2}\right) w' + \frac{z^4}{2} w''.$$

This is equivalent to $z^4 w'' + (2z^3 - z^2)w' - 2w = 0$. □

(3)

Proof. Let $w = c_0 w_1 + c_1 w_2$. Then we note $w_1' = \frac{a}{z^2} w_2$ and $w_2' = -\frac{a}{z^2} w_1$. So $w' = c_0 \cdot \frac{a}{z^2} w_2 - c_1 \cdot \frac{a}{z^2} w_1$. Multiplying both sides by z^2, we get $z^2 w' = c_0 a w_2 - c_1 a w_1$ (*). Differentiating both sides again, we have $2zw' + z^2 w'' = c_0 a \left(-\frac{a}{z^2}\right) w_1 - c_1 a \left(\frac{a}{z^2}\right) w_2$, i.e. $2z^3 w' + z^4 w'' = -c_0 a^2 w_1 - c_1 a^2 w_2$ (**). Combining (*) and (**), we can get

$$\begin{cases} ac_1 z^2 w' + c_0(2z^3 w' + z^4 w'') = -(c_1^2 + c_0^2)a^2 w_1 \\ ac_0 z^2 w' - c_1(2z^3 w' + z^4 w'') = (c_0^2 + c_1^2)a^2 w_2. \end{cases}$$

Using the fact $w = c_0 w_1 + c_1 w_2$, we can easily get $z^4 w'' + 2z^3 w' + a^2 w = 0$. □

(4)

Proof. Let $w = c_0 w_1 + c_1 w_2$. Then $(z^2 - 1)w = c_0 z^2 + c_1 z$. Differentiating both sides of the equation gives $2zw + (z^2 - 1)w' = 2c_0 z + c_1$. Differentiate again, we get $2w + 2zw' + 2zw' + (z^2 - 1)w'' = 2c_0$. Solving for c_0, c_1, and plug the expressions into the equation $w = c_0 w_1 + c_1 w_2$, we get

$$z^2(z^2 - 1)w'' + 2z(z^2 + 1)w' - 2w = 0.$$

□

2. (1)

Proof. $p(z) = z^2$ and $q(z) = 0$ are both analytic. So the solution $w(z)$ is analytic and assumes the form $w(z) = \sum_{k=0}^{\infty} a_k z^k$. Then

$$w'' - z^2 w = \sum_{k=2}^{\infty} a_k \cdot k(k-1)z^{k-2} - \sum_{k=0}^{\infty} a_k z^{k+2} = 2a_2 + 6a_3 z + \sum_{k=2}^{\infty} [a_{k+2} \cdot (k+2)(k+1) - a_{k-2}] z^k = 0.$$

Clearly, $a_0 = w(0)$ and $a_1 = w'(0)$. By the uniqueness of power series representation of an analytic function, we have $a_2 = a_3 = 0$, and $a_{k+2} = \frac{a_{k-2}}{(k+2)(k+1)}$. So

$$a_{4k} = \frac{a_{4(k-1)}}{4k \cdot (4k-1)} = \frac{a_0}{4k \cdot (4k-1) \cdot \cdots \cdot 4 \cdot 3} = \frac{a_0}{4^{2k} \cdot k! \cdot (k - \frac{1}{4}) \cdots (1 - \frac{1}{4})} = \frac{a_0 \Gamma(\frac{3}{4})}{2^{4k} \cdot k! \cdot \Gamma(k + \frac{3}{4})}$$

and

$$a_{4k+1} = \frac{a_{4(k-1)+1}}{(4k+1) \cdot 4k} = \frac{a_1}{(4k+1) \cdot 4k \cdot \cdots \cdot 5 \cdot 4} = \frac{a_1}{4^{2k} \cdot k! \cdot (k + \frac{1}{4}) \cdot [(k-1) + \frac{1}{4}] \cdots (1 + \frac{1}{4})} = \frac{a_1 \Gamma(\frac{5}{4})}{2^{4k} \cdot k! \cdot \Gamma(k + \frac{5}{4})}.$$

Let $w_1(z) = \sum_{n=0}^{\infty} \frac{\Gamma(\frac{3}{4})}{n! \cdot \Gamma(n + \frac{3}{4})} \left(\frac{z}{2}\right)^{4n}$ and $w_2(z) = \sum_{n=0}^{\infty} \frac{2\Gamma(\frac{5}{4})}{n! \cdot \Gamma(n + \frac{5}{4})} \left(\frac{z}{2}\right)^{4n+1}$. Then $w(z) = a_0 w_1(z) + a_1 w_2(z)$. □

(2)

Proof. $p(z) = z$ and $q(z) = 0$ are both analytic. So the solution $w(z)$ is analytic and assumes the form $w(z) = \sum_{k=0}^{\infty} a_k z^k$. Then

$$\begin{aligned} w'' - zw &= \sum_{k=2}^{\infty} a_k \cdot k(k-1)z^{k-2} - \sum_{k=0}^{\infty} a_k z^{k+1} \\ &= \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)z^k - \sum_{k=1}^{\infty} a_{k-1}z^k \\ &= 2a_2 + \sum_{k=1}^{\infty}[a_{k+2}(k+2)(k+1) - a_{k-1}]z^k \\ &= 0. \end{aligned}$$

Clearly, $a_0 = w(0)$ and $a_1 = w'(0)$. By the uniqueness of power series representation of an analytic function, we have $a_2 = 0$, and $a_{k+2} = \frac{a_{k-1}}{(k+2)(k+1)}$. So

$$a_{3k} = \frac{a_{3(k-1)}}{3k \cdot (3k-1)} = \cdots = \frac{a_0}{3k \cdot (3k-1) \cdot \cdots \cdot 3 \cdot 2} = \frac{a_0}{3^{2k} \cdot k! \cdot (k - \frac{1}{3}) \cdots (1 - \frac{1}{3})} = \frac{a_0 \Gamma(\frac{2}{3})}{3^{2k} \cdot k! \cdot \Gamma(k + \frac{2}{3})}$$

and

$$a_{3k+1} = \frac{a_{3(k-1)+1}}{(3k+1) \cdot 3k} = \cdots = \frac{a_1}{(3k+1) \cdot 3k \cdot \cdots \cdot 4 \cdot 3} = \frac{a_1}{3^{2k} \cdot k! \cdot (k + \frac{1}{3}) \cdots (1 + \frac{1}{3})} = \frac{a_1 \Gamma(\frac{4}{3})}{3^{2k} \cdot k! \cdot \Gamma(k + \frac{4}{3})}.$$

Let $w_1(z) = \sum_{n=0}^{\infty} \frac{\Gamma(\frac{2}{3})}{n! \cdot \Gamma(n+\frac{2}{3})} \frac{z^{3n}}{3^{2n}}$ and $w_2(z) = \sum_{n=0}^{\infty} \frac{\Gamma(\frac{4}{3})}{n! \cdot \Gamma(n+\frac{4}{3})} \frac{z^{3n+1}}{3^{2n}}$. Then $w(z) = a_0 w_1(z) + a_1 w_2(z)$. □

(3)

Proof. The equation can be written as $w'' + p(z)w + q(z) = 0$ with $p(z) = \frac{z}{z^2-1}$ and $q(z) = -\frac{1}{z^2-1}$. So ± 1 are two singularities of the equation and the equation has an analytic solution in a neighborhood of 0 (e.g. $D(0,1) := \{z : |z| < 1\}$). Suppose $w = \sum_{k=0}^{\infty} a_k z^k$. Then

$$(z^2-1)w'' + zw' - w = \sum_{k=2}^{\infty}[(k-1)(k+1)a_k - (k+1)(k+2)a_{k+2}]z^k - (a_0 + 2a_2) - 6a_3 z = 0.$$

Therefore, $a_0 = w(0)$, $a_1 = w'(0)$, $a_2 = -\frac{w(0)}{2}$, $a_3 = 0$, and for $k \geq 2$,

$$a_{k+2} = \frac{(k-1)a_k}{k+2}.$$

From $a_3 = 0$, we conclude $a_{2k+1} = 0$ for $k \geq 1$. Moreover

$$a_{2k} = \frac{2k-3}{2k} a_{2k-2} = \frac{2k-3}{2k} \cdot \frac{2k-5}{2k-2} a_{2k-4} = \cdots = \frac{2^k[(k-1) - \frac{1}{2}][(k-2) - \frac{1}{2}] \cdots (0 - \frac{1}{2})}{2^k \cdot k!} a_0 = \frac{\Gamma(k - \frac{1}{2})}{k! \cdot \Gamma(-\frac{1}{2})} a_0.$$

Let $w_1(z) = \sum_{k=0}^{\infty} \frac{\Gamma(k-\frac{1}{2})}{k! \cdot \Gamma(-\frac{1}{2})} z^{2k}$ and $w_2(z) = z$, we get $w(z) = a_0 w_1(z) + a_1 w_2(z)$.

Remark 1. *Note $w_1(z)$ is the Taylor series of $\sqrt{1-z^2}$.*

□

(4)

Proof. The equation can be written as $w'' + p(z)w + q(z) = 0$ with $p(z) = \frac{2(1+2z)}{1+z+z^2}$ and $q(z) = \frac{2}{1+z+z^2}$. So $\frac{-1\pm\sqrt{3}i}{2}$ are two singularities of the equation and the equation has an analytic solution in $D(0,1)$. Suppose $w = \sum_{k=0}^{\infty} a_k z^k$. Then

$$(1+z+z^2)w''+2(1+2z)w'+zw = \sum_{k=2}^{\infty}(k+1)(k+2)(a_k+a_{k+1}+a_{k+2})z^k+2(a_0+a_1+a_2)+6(a_1+a_2+a_3)z = 0.$$

Therefore, we have

$$\begin{cases} a_0 + a_1 + a_2 = 0 \\ a_1 + a_2 + a_3 = 0 \\ a_k + a_{k+1} + a_{k+2} = 0 \ (k \geq 2). \end{cases}$$

Solving it gives us $a_{3k} = a_0$, $a_{3k+1} = a_1$, and $a_{3k+2} = -(a_0 + a_1)$. Plugging in the values of a_k's, we get after simplification $w(z) = a_0 w_1(z) + a_1 w_2(z)$, where $w_1(z) = \frac{1+z}{1+z+z^2}$ and $w_2(z) = \frac{z}{1+z+z^2}$. Equivalently, we can choose another basis with two linearly independent solutions: $\widetilde{w}_1 = \frac{1}{1+z+z^2}$ and $\widetilde{w}_2(z) = \frac{z}{1+z+z^2}$.

Remark 2. *We can solve the problem more directly once we note*

$$\frac{d^2}{dz^2}[(1+z+z^2)w(z)] = (1+z+z^2)w''(z) + 2(1+2z)w'(z) + 2w(z).$$

So $(1+z+z^2)w(z) = a_0 + a_1 z$, *i.e.*

$$w(z) = \frac{a_0}{1+z+z^2} + \frac{a_1 z}{1+z+z^2}.$$

□

3. (1)

Proof. The equation can be transformed to $w'' + p(z)w' + q(z)w = 0$, where $p(z) = \frac{1-3z}{z(1-z)}$ and $q(z) = -\frac{1+z}{z^2(1-z)}$. So 0 is the singularity of the equation. Since $zp(z) = \frac{1-3z}{1-z}$ and $z^2q(z) = -\frac{1+z}{1-z}$ are analytic in $0 < |z| < 1$, the equation has two regular solutions in $0 < |z| < 1$:

$$\begin{cases} w_1(z) = z^{\rho_1} \sum_{k=0}^{\infty} c_k z^k \ (c_0 \neq 0) \\ w_2(z) = g w_1(z) \ln z + z^{\rho_2} \sum_{k=0}^{\infty} d_k z^k \ (g \neq 0 \text{ or } d_0 \neq 0) \end{cases}$$

for some constants g, ρ_1 and ρ_2. ρ_1 and ρ_2 satisfy the index equation $\rho(\rho-1) + a_0\rho + b_0 = 0$, where $a_0 = \lim_{z\to 0} zp(z) = 1$ and $b_0 = \lim_{z\to 0} z^2 q(z) = -1$. So the equation for ρ becomes $\rho^2 = 1$ and ρ is therefore 1. Suppose $w(z)$ has the form of $z^\rho \sum_{n=0}^{\infty} c_n z^n$, then we can get the following the recursive relation

$$[(n+\rho)(n+\rho-1) + a_0(n+\rho) + b_0]c_n + \sum_{l=0}^{n-1}[a_{n-l}(l+\rho) + b_{n-l}]c_l = 0 \ (n \geq 1),$$

where a_n's and b_n's come from the Laurent series of $p(z) = \sum_{l=0}^{\infty} a_l z^{l-1}$ and $q(z) = \sum_{l=0}^{\infty} b_l z^{l-2}$, respectively. We then find the Laurent series expansion of $p(z)$ and $q(z)$ as follows:

$$p(z) = \frac{1}{z} - 2\sum_{l=0}^{\infty} z^l, \ q(z) = -\frac{1}{z^2} - 2\sum_{l=1}^{\infty} z^{l-2}.$$

Therefore the recursive relation can be simplified to

$$[(n+1)n + (n+1) - 1]c_n + \sum_{l=0}^{n-1}[-2(l+1) + (-2)]c_l = 0.$$

Define $\xi_n = (n+2)c_n$ $(n \geq 2)$. Then this relation can be further simplified to

$$\xi_n = 2 \cdot \frac{\sum_{k=0}^{n-1} \xi_k}{n}.$$

It's easy to see by induction that $\xi_n = (n+1)\xi_0 = (n+1)c_0$. Therefore

$$w(z) = z^\rho \sum_{k=0}^{\infty} c_k z^k = c_0 \sum_{k=0}^{\infty} \frac{k+1}{k+2} z^{k+1} = c_0 \sum_{k=0}^{\infty} z^{k+1} - c_0 \sum_{k=0}^{\infty} \frac{z^{k+1}}{k+2} = \frac{c_0 z}{1-z} - \frac{c_0}{z}[-\ln(1-z) - z] = c_0 w_1(z),$$

where $w_1(z) = \frac{1}{1-z} + \frac{\ln(1-z)}{z}$. From formula (6.27b), we can conjecture $w_2(z) = \frac{1}{z}$. Then it's easy to verify this conjecture is indeed true. □

(2)

Proof. The equation can be transformed to $w'' + p(z)w' + q(z)w = 0$, where $p(z) = -\frac{5}{3z}$ and $q(z) = \frac{7}{9z^2} + 4z^2$. So 0 is the singularity of the equation. Since $zp(z) = -\frac{5}{3}$ and $z^2 q(z) = \frac{7}{9} + 4z^4$ are analytic in $0 < |z|$, the equation has two regular solutions in $0 < |z|$:

$$\begin{cases} w_1(z) = z^{\rho_1} \sum_{k=0}^{\infty} c_k z^k, \ (c_0 \neq 0) \\ w_2(z) = g w_1(z) \ln z + z^{\rho_2} \sum_{k=0}^{\infty} d_k z^k, \ (g \neq 0 \text{ or } d_0 \neq 0) \end{cases}$$

for some constants g, ρ_1 and ρ_2. ρ_1 and ρ_2 satisfy the index equation

$$\rho(\rho - 1) + a_0 \rho + b_0 = 0,$$

where $a_0 = \lim_{z\to\infty} zp(z) = -\frac{5}{3}$ and $b_0 = \lim_{z\to 0} z^2 q(z) = \frac{7}{9}$. Solving the above equation gives $\rho_1 = \frac{7}{3}$ and $\rho_2 = \frac{1}{3}$. We note the Laurent series expansion of $p(z)$ and $q(z)$ are, respectively, $-\frac{5}{3z}$ and $\frac{7}{9z^2} + 4z^2$. So $a_0 = -\frac{5}{3}$ and $a_n = 0$ for $n \geq 1$; $b_0 = \frac{7}{9}$, $b_4 = 4$, and $b_n = 0$ for $n \geq 1$ and $n \neq 4$. Therefore, the recursion equation

$$[(n+\rho)(n+\rho-1) + a_0(n+\rho) + b_0]c_n + \sum_{l=0}^{n-1} [a_{n-l}(l+\rho) + b_{n-l}]c_l = 0 \ (n \geq 1)$$

is simplified to

$$\left[(n+\rho)^2 - \frac{8}{3}(n+\rho) + \frac{7}{9}\right] c_n + 1_{\{n\geq 4\}} 4c_{n-4} = 0 \ (n \geq 1)$$

and we can conclude

$$c_n = \begin{cases} 0 & \text{if } n = 1,2,3 \\ -\frac{4c_{n-4}}{(n+\rho)(n+\rho-\frac{8}{3}) + \frac{7}{9}} & \text{if } n \geq 4. \end{cases}$$

Now let $\rho = \rho_1 = \frac{7}{3}$. Then for $n \geq 4$,

$$c_n = -\frac{4c_{n-4}}{n(n+2)}$$

Define $\xi_k = c_{4k}$ $(k \geq 0)$. Then for $k \geq 1$,

$$\xi_k = -\frac{4\xi_{k-1}}{4k \cdot (4k+2)} = -\frac{\xi_{k-1}}{(2k)\cdot(2k+1)}.$$

By induction, it's easy to see $\xi_k = \frac{(-1)^k}{(2k+1)!}\xi_0 = \frac{(-1)^k}{(2k+1)!}c_0$. Therefore

$$w(z) = c_0 z^{\frac{7}{3}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{4k} = c_0 z^{\frac{1}{3}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{4k+2} = c_0 w_1(z),$$

where $w_1(z) = z^{\frac{1}{3}} \sin(z^2)$. Similarly, by choosing $\rho = \rho_2 = \frac{1}{3}$, the recursion equation can be written as $(n^2 - 2n)c_n + 4c_{n-4} = 0$, which leads to the solution $c_{4k} = \frac{(-1)^k}{(2k)!}c_0$. Therefore, the other solution is

$$w_2(z) = z^{\frac{1}{3}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^{4k} = z^{\frac{1}{3}} \cos(z^2).$$

□

(3)

Proof. The equation can be transformed to $w''(z) - w'(z) + \frac{1}{z}w(z) = 0$. Let $p(z) = -1$ and $q(z) = \frac{1}{z}$. Then $zp(z) = -z$ and $z^2q(z) = z$ are both analytic in $|z| > 0$. By Theorem 6.3, the equation has two regular solutions in $|z| > 0$:

$$\begin{cases} w_1(z) = z^{\rho_1} \sum_{k=0}^{\infty} c_k z^k & (c_0 \neq 0) \\ w_2(z) = g w_1(z) \ln z + z^{\rho_2} \sum_{k=0}^{\infty} d_k z^k & (g \neq 0 \text{ or } d_0 \neq 0) \end{cases}$$

for some constants g, ρ_1 and ρ_2. ρ_1 and ρ_2 satisfy the index equation

$$\rho(\rho - 1) + a_0\rho + b_0 = 0,$$

where $a_0 = \lim_{z\to 0} zp(z) = 0$ and $b_0 = \lim_{z\to 0} z^2q(z) = 0$. So $\rho_1 = 1$ and $\rho_2 = 0$.

Suppose $w(z)$ has the form $z^\rho \sum_{n=0}^{\infty} c_n z^n$. We can get the following recursion equation

$$[(n+\rho)(n+\rho-1) + a_0(n+\rho) + b_0]c_n + \sum_{l=0}^{n-1} [a_{n-l}(l+\rho) + b_{n-l}]c_l = 0 \ (n \geq 1).$$

Since $a_0 = b_0 = 0$, $a_1 = -1$, $a_n = 0$ $(n \geq 2)$, $b_1 = 1$, $b_n = 0$ $(n \geq 2)$, the above equation can be further simplified: if $n = 1$, the equation becomes

$$(\rho + 1)\rho \cdot c_1 = 0;$$

if $n \geq 2$, the equation becomes

$$(n+\rho)(n+\rho-1)c_n + (-n+2-\rho)c_{n-1} = 0.$$

We first let $\rho = \rho_1 = 1$. Then $c_1 = 0$ and $(n+1)nc_n = (n-1)c_{n-1}$ for $n \geq 2$. Therefore, $c_n = 0$ for $n \geq 1$, and one solution of the equation is

$$w(z) = z^{\rho_1} \sum_{n=0}^{\infty} c_n z^n = c_0 z.$$

So we can let $w_1(z) = z$. We then let $\rho = \rho_2 = 0$. Then c_1 can be any number and $n(n-1)c_n = (n-2)c_{n-1}$ for $n \geq 2$. This implies $c_n = 0$ for $n \geq 2$. The corresponding solution is therefore

$$z^{\rho_2} \sum_{n=0}^{\infty} c_n z^n = c_0 + c_1 z.$$

So we get back to the same solution $w(z) = z$. This means we have to try the other form of the solution

$$w_2(z) = g w_1(z) \ln z + z^{\rho_2} \sum_{n=0}^{\infty} d_n z^n = g z \ln z + \sum_{n=0}^{\infty} d_n z^n,$$

where g is a constant. Note

$$w_2'(z) = g \ln z + g + \sum_{n=1}^{\infty} d_n n z^{n-1} \text{ and } w_2''(z) = \frac{g}{z} + \sum_{n=2}^{\infty} d_n n(n-1) z^{n-2}.$$

So

$$
\begin{aligned}
& w_2''(z)+p(z)w_2'(z)+q(z)w_2(z) \\
= & \left[\frac{g}{z}+\sum_{n=2}^{\infty} d_n n(n-1)z^{n-2}\right]-\left[g\ln z+g+\sum_{n=1}^{\infty} d_n n z^{n-1}\right]+\left[g\ln z+\sum_{n=0}^{\infty} d_n z^{n-1}\right] \\
= & \frac{g+d_0}{z}+(2d_2-g)+\sum_{n=1}^{\infty}[d_{n+2}(n+2)(n+1)-d_{n+1}n]z^n.
\end{aligned}
$$

Therefore the necessary and sufficient condition for $w_2''(z)+p(z)w_2'(z)+q(z)w_2(z)=0$ to hold is

$$
\begin{cases}
g+d_0=0 \\
g=2d_2 \\
d_{n+2}(n+2)(n+1)=d_{n+1}n \ (n\geq 1).
\end{cases}
$$

We have $g=-d_0$ and $d_2=-\frac{1}{2}d_0$. By defining $\xi_n=(n-1)d_n$ $(n\geq 2)$ and working by induction, it's easy to deduce $d_n=\frac{1}{(n-1)\cdot n!}$ for $n\geq 2$. So we have

$$
w_2(z)=-d_0 z\ln z+d_0+d_1 z-\sum_{n=2}^{\infty}\frac{d_0}{(n-1)\cdot n!}z^n.
$$

Note $w(z)=z$ is already a solution, so we can let $w_2(z)=z\ln z-1+\sum_{n=2}^{\infty}\frac{z^n}{n!(n-1)}$. □

(4)

Proof. The equation can be transformed to $w''(z)+p(z)w'(z)+q(z)w(z)=0$, where $p(z)=1-\frac{1}{z}$ and $q(z)=\frac{1}{z}$. Since $zp(z)=z-1$ and $z^2q(z)=z$ are both analytic in $|z|>0$, the equation has two regular solution in $|z|>0$:

$$
\begin{cases}
w_1(z)=z^{\rho_1}\sum_{k=0}^{\infty}c_k z^k & (c_0\neq 0) \\
w_2(z)=gw_1(z)\ln z+z^{\rho_2}\sum_{k=0}^{\infty}d_k z^k & (g\neq 0 \text{ or } d_0\neq 0)
\end{cases}
$$

for some constants g, ρ_1 and ρ_2. ρ_1 and ρ_2 satisfy the index equation

$$
\rho(\rho-1)+a_0\rho+b_0=0,
$$

where $a_0=\lim_{z\to 0} zp(z)=-1$ and $b_0=\lim_{z\to 0} z^2q(z)=0$. So $\rho_1=2$ and $\rho_2=0$.

Suppose $w(z)$ has the form $z^\rho\sum_{n=0}^{\infty}c_n z^n$. We can get the following recursion equation

$$
[(n+\rho)(n+\rho-1)+a_0(n+\rho)+b_0]c_n+\sum_{l=0}^{n-1}[a_{n-l}(l+\rho)+b_{n-l}]c_l=0 \ (n\geq 1).
$$

Since $a_0=-1$, $a_1=1$, and $a_n=0$ $(n\geq 2)$; $b_0=0$, $b_1=1$, and $b_n=0$ $(n\geq 2)$, the above recursion equation can be further simplified to $(n+\rho-2)c_n+c_{n-1}=0$.

We first let $\rho=\rho_1=2$. Then it's easy to see $c_n=\frac{(-1)^n}{n!}c_0$. So we can let $w_1(z)=z^2\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}z^n=z^2e^{-z}$. We then let $\rho=\rho_2=0$, and the recursion equation becomes $(n-2)c_n+c_{n-1}=0$ $(n\geq 1)$. It's easy to see $c_0=c_1=0$ and $c_n=\frac{(-1)^n}{(n-2)!}c_2$ for $n\geq 2$. Plugging these values into the formula $w(z)=z^{\rho_2}\sum_{n=0}^{\infty}c_n z^n$, we get $w(z)=c_2z^2e^{-z}$. This is the same as the first solution, so we have to try the other form of the solution

$$
w_2(z)=gw_1(z)\ln z+z^{\rho_2}\sum_{n=0}^{\infty}d_n z^n=gz^2e^{-z}\ln z+\sum_{n=0}^{\infty}d_n z^n,
$$

where g is a constant. Note

$$w_2'(z) = ge^{-z}(2z\ln z - z^2\ln z + z) + \sum_{n=1}^{\infty} d_n n z^{n-1}$$

and

$$w_2''(z) = ge^{-z}[3 - 2z + (2 - 4z + z^2)\ln z] + \sum_{n=2}^{\infty}(n-1)nd_n z^{n-2}.$$

So

$$w''(z) + p(z)w'(z) + q(z)w(z) = -ge^{-z}(-2+z) + \frac{d_0 - d_1}{z} + \sum_{n=0}^{\infty}(n+2)(nd_{n+2} + d_{n+1})z^n.$$

Set $w''(z)+p(z)w'(z)+q(z)w(z) = 0$. Then it's easy to see $g = 0$ and by induction we must have $d_0 = d_1 = 0$, $d_{n+1} = \frac{(-1)^{n-1}}{(n-1)!}d_2$ for $n \geq 2$. This leads us back to the first solution.

Therefore, we apply Liouville's formula, formula (6.30), to get the second solution: (note $p(z) = 1 - \frac{1}{z}$)

$$\begin{aligned}
w_2(z) &= w_1(z)\int^z \left\{\frac{1}{[w_1(\eta)]^2}\exp\left[-\int^\eta p(\xi)d\xi\right]\right\}d\eta \\
&= w_1(z)\int^z \left\{\frac{1}{[w_1(\eta)]^2}\exp\left[-\eta + \ln\eta\right]\right\}d\eta \\
&= z^2e^{-z}\int^z \frac{\eta e^{-\eta}}{\eta^4 e^{-2\eta}}d\eta \\
&= z^2e^{-z}\int^z \frac{e^\eta}{\eta^3}d\eta.
\end{aligned}$$

Denote by $Ei(z)$ the exponential integral function $Ei(z) = \int_{-\infty}^z \frac{e^\xi}{\xi}d\xi$. Then by integration-by-part formula, we have

$$w_2(z) = z^2e^{-z}\cdot\frac{-e^z(1+z) + z^2Ei(i)}{2z^2} = \frac{1}{2}[-(1+z) + z^2e^{-z}Ei(z)].$$

Remark 3. *Verify the $w_2(z)$ represented in this form is the same as the $w_2(z)$ given by the textbook's solution.*

□

4.

Proof. In the given equation, we have $p(z) = \frac{2}{z}$ and $q(z) = m^2$. Since $zp(z) = 2$ and $z^2q(z) = m^2z^2$ are both analytic in $|z| > 0$, the equation has two regular solution in $|z| > 0$:

$$\begin{cases} w_1(z) = z^{\rho_1}\sum_{k=0}^{\infty} c_k z^k & (c_0 \neq 0) \\ w_2(z) = gw_1(z)\ln z + z^{\rho_2}\sum_{k=0}^{\infty} d_k z^k & (g \neq 0 \text{ or } d_0 \neq 0) \end{cases}$$

for some constants g, ρ_1 and ρ_2. ρ_1 and ρ_2 satisfy the index equation

$$\rho(\rho-1) + a_0\rho + b_0 = 0,$$

where $a_0 = \lim_{z\to 0} zp(z) = 2$ and $b_0 = \lim_{z\to 0} z^2q(z) = 0$. So $\rho_1 = 0$ and $\rho_2 = -1$.

Let $w_1(z) = z^{\rho_1}\sum_{n=0}^{\infty} c_n z^n = \sum_{n=0}^{\infty} c_n z^n$. Then

$$w_1'(z) = \sum_{n=0}^{\infty} c_{n+1}(n+1)z^n \quad\text{and}\quad w_1''(z) = \sum_{n=0}^{\infty} c_{n+2}(n+2)(n+1)z^n.$$

Therefore

$$w''(z) + p(z)w'(z) + q(z)w(z) = \sum_{n=0}^{\infty}[c_{n+2}(n+2)(n+3) + m^2c_n]z^n + \frac{2c_1}{z} = 0.$$

This implies

$$\begin{cases} c_1 = 0 \\ c_{n+2}(n+2)(n+3) + m^2 c_n = 0, \quad n \ge 0. \end{cases}$$

If $m = 0$, the equation has only a constant solution $w(z) = c_0$. If $m \neq 0$, we must have

$$c_{n+2} = \frac{-m^2}{(n+2)(n+3)} c_n, \ n \ge 0.$$

Define $\xi_k = c_{2k}$ $(k \ge 0)$. Then the above relation can be written as

$$\begin{aligned} \xi_{k+1} &= c_{2k+2} = \frac{-m^2}{(2k+3)(2k+2)} c_{2k} = \frac{-m^2}{(2k+3)(2k+2)} \xi_k \\ &= \cdots \\ &= \frac{-m^2}{(2k+3)(2k+2)} \cdot \frac{-m^2}{(2k+1)\cdot 2k} \cdots \frac{-m^2}{3\cdot 2} c_0 \\ &= \frac{(-1)^{k+1} m^{2(k+1)}}{(2(k+1)+1)!} c_0. \end{aligned}$$

Therefore

$$w_1(z) = \sum_{n=0}^{\infty} c_n z^n = c_0 + c_0 \sum_{k=1}^{\infty} \frac{(-1)^k m^{2k}}{(2k+1)!} z^{2k} = \frac{c_0}{mz}\left[mz + \sum_{k=1}^{\infty} \frac{(-1)^k m^{2k+1}}{(2k+1)!} z^{2k+1}\right] = c_0 \frac{\sin mz}{mz}.$$

Let $w_2(z) = z^{\rho_2} \sum_{n=0}^{\infty} d_n z^n = \frac{d_0}{z} + \sum_{n=0}^{\infty} d_{n+1} z^n$. So $w_2'(z) = -\frac{d_0}{z^2} + \sum_{n=1}^{\infty} d_{n+1} n z^{n-1}$ and $w_2''(z) = \frac{2d_0}{z^3} + \sum_{n=2}^{\infty} d_{n+1} n(n-1) z^{n-2}$. Therefore

$$w_2''(z) + p(z) w_2'(z) + q(z) w_2(z) = \frac{2d_2 + m^2}{z} + \sum_{n=0}^{\infty} [(n+2)(n+3) d_{n+3} + m^2 d_{n+1}] z^n.$$

Then we have the equations for d_n's:

$$\begin{cases} 2d_2 + m^2 = 0 \\ (n+2)(n+3) d_{n+3} + m^2 d_{n+1} = 0, \ n \ge 0. \end{cases}$$

Following a procedure similar to that of the first solution, we can easily find d_n's and prove the second solution is $w_2(z) = \frac{\cos mz}{mz}$. □

5.

Proof. In the given equation, we have $p(z) = \frac{1}{z}$ and $q(z) = -m^2$. Since $zp(z) = 1$ and $z^2 q(z) = -m^2 z^2$ are both analytic in $|z| > 0$, the equation has two regular solution in $|z| > 0$:

$$\begin{cases} w_1(z) = z^{\rho_1} \sum_{k=0}^{\infty} c_k z^k & (c_0 \neq 0) \\ w_2(z) = g w_1(z) \ln z + z^{\rho_2} \sum_{k=0}^{\infty} d_k z^k & (g \neq 0 \text{ or } d_0 \neq 0) \end{cases}$$

for some constants g, ρ_1 and ρ_2. ρ_1 and ρ_2 satisfy the index equation

$$\rho(\rho - 1) + a_0 \rho + b_0 = 0,$$

where $a_0 = \lim_{z\to 0} zp(z) = 1$ and $b_0 = \lim_{z \to 0} z^2 q(z) = 0$. So $\rho_1 = \rho_2 = 0$.

Let $w(z) = z^{\rho} \sum_{n=0}^{\infty} c_n z^n = \sum_{n=0}^{\infty} c_n z^n$. Then

$$w''(z) + p(z) w'(z) + q(z) w(z) = \frac{c_1}{z} + \sum_{n=0}^{\infty} [c_{n+2}(n+2)^2 - m^2 c_n] z^n.$$

So we have the equations for c_n's:

$$\begin{cases} c_1 = 0 \\ c_{n+2} = \frac{m^2}{(n+2)^2} c_n, \ n \geq 0. \end{cases}$$

Working by induction, it's easy to see ($k \geq 0$)

$$c_n = \begin{cases} 0, & n = 2k+1 \\ \frac{\left(\frac{m}{2}\right)^{2k}}{(k!)^2} c_0, & n = 2k. \end{cases}$$

Therefore, one solution of the ordinary differential equation is

$$w_1(z) = \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\frac{mz}{2}\right)^{2k} = I_0(mz),$$

where $I_\alpha(z)$ is the modified Bessel function

$$I_\alpha(z) = \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(k+\alpha+1)} \left(\frac{z}{2}\right)^{2k+\alpha}.$$

Note the series representation for Bessel function J_n is

$$J_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{2k+n}}{k!(k+n)!},$$

$w_1(z)$ can also be written as $J_0(imz)$.

To get the other solution, we let $w(z) = gI_0(mz)\ln z + \sum_{n=0}^{\infty} d_n z^n$, where g is a constant. Plug this representation into the equation $w''(z) + p(z)w'(z) + q(z)w(z) = 0$, we can get equations for g and d_n's.

Now the computation becomes really messy, so we omit the details for this version. Mathematica command **DSolve[w"[z]+w'[z]/z - m^ 2w[z] == 0, w[z], z]** gives the two solutions as $J_0(-imz) = J_0(imz)$ and $Y_0(-imz)$. Here $Y_\alpha(z)$ is Bessel function of the second kind and is defined as

$$Y_\alpha(z) = \frac{J_\alpha(z)\cos(\alpha\pi) - J_{-\alpha}(z)}{\sin(\alpha z)}.$$

Remark 4. *Verify $w_2(z)$ represented in this form is the same as the $w_2(z)$ given by the textbook's solution.*

□

7 Residue Theorem and Its Applications

7.1 Exercises in the text

7.1.

Proof. Suppose $f(z)$ has Laurent series $\sum_{n=-\infty}^{\infty} a_n z^n$ in a neighborhood of 0. Then

$$f(z) = \frac{1}{2}[f(z) + f(-z)] = \frac{1}{2}\sum_{n=-\infty}^{\infty} a_n z^n [1 + (-1)^n] = \sum_{k=-\infty}^{\infty} a_{2k} z^{2k}.$$

So for ρ sufficiently small, $\mathrm{Res}(f, 0) = \sum_{k=-\infty}^{\infty} \frac{1}{2\pi i}\int_{|z|=\rho} z^{2k} dz = 0$, where the last equality is due to Cauchy's integral formula. □

7.2. $f(z)$ can be written as $z^n g(z)$ in a neighborhood of 0, where $g(z)$ is analytic near 0 and $g(0) \neq 0$. Without loss of generality, we assume $n \geq 1$. Otherwise, all the residues below equal to 0.

(1)

Proof.

$$\frac{f'(z)}{f(z)} = \frac{nz^{n-1}g(z) + z^n g'(z)}{z^n g(z)} = \frac{n}{z} + \frac{g'(z)}{g(z)}.$$

So $\mathrm{Res}(f'/f, 0) = n$. □

(2)

Proof. If $n = 1$,

$$\frac{f''(z)}{f(z)} = \frac{[g(z) + zg'(z)]'}{f(z)} = \frac{2g'(z) + zg''(z)}{zg(z)} = \frac{2}{z} \cdot \frac{g'(z)}{g(z)} + \frac{g''(z)}{g(z)}.$$

So $\mathrm{Res}(f''/f, 0) = 2g'(0)/g(0)$. If $n \geq 2$,

$$\frac{f''(z)}{f(z)} = \frac{n(n-1)z^{n-2}g(z) + 2nz^{n-1}g'(z) + z^n g''(z)}{f(z)} = \frac{n(n-1)}{z^2} + \frac{2n}{z} \cdot \frac{g'(z)}{g(z)} + \frac{g''(z)}{g(z)}.$$

So $\mathrm{Res}(f''/f, 0) = 2ng'(0)/g(0)$. Combined, we conclude $\mathrm{Res}(f''/f, 0) = 2ng'(0)/g(0)$. □

(3)

Proof. If $n = 1$,

$$\frac{f''(z)}{f'(z)} = \frac{2g'(z) + zg''(z)}{g(z) + zg'(z)}$$

is analytic near 0. So its residue is equal to 0. If $n \geq 2$,

$$\frac{f''(z)}{f'(z)} = \frac{n(n-1)z^{n-2}g(z) + 2nz^{n-1}g'(z) + z^n g''(z)}{nz^{n-1}g(z) + z^n g'(z)} = \frac{1}{z} \cdot \frac{n(n-1)g(z) + 2nzg'(z) + z^2 g''(z)}{ng(z) + zg'(z)}.$$

Note the second term in the product is analytic near 0, so $\mathrm{Res}(f''/f', 0) = \left.\frac{n(n-1)g(z)+2nzg'(z)+z^2g''(z)}{ng(z)+zg'(z)}\right|_{z=0} = n-1$. Combined, we conclude the residue is equal to $n-1$. □

(4)

Proof. If $n = 1$,

$$\frac{(n-1)f'(z) - zf''(z)}{f(z)} = -z\frac{f''(z)}{f(z)} = -z \cdot \frac{2g'(z) + zg''(z)}{g(z) + zg'(z)}$$

is analytic near 0, so its residue at 0 is equal to 0. If $n \geq 2$,

$$\frac{(n-1)f'(z) - zf''(z)}{f(z)} = (n-1)\frac{f'(z)}{f(z)} - z\frac{f''(z)}{f(z)} = n(n-1) - n(n-1) = 0,$$

where the second to last equality is due to part (1) and (2). □

7.3. $f(z)$ can be written as $z^{-n}g(z)$, where g is analytic near 0 and $g(0) \neq 0$. Without loss of generality, we assume $n \geq 1$. Otherwise, all the residues below are equal to 0.

(1)

Proof.

$$\frac{f'(z)}{f(z)} = \frac{-nz^{-n-1}g(z) + z^{-n}g'(z)}{z^{-n}g(z)} = \frac{-ng(z) + zg'(z)}{zg(z)} = -\frac{n}{z} + \frac{g'(z)}{g(z)}.$$

So $\mathrm{Res}(f'/f, 0) = -n$. □

(2)

Proof.

$$\frac{f''(z)}{f(z)} = \frac{n(n+1)z^{-n-2}g(z) - 2nz^{-n-1}g'(z) + z^{-n}g''(z)}{z^{-n}g(z)} = \frac{n(n+1)}{z^2} - \frac{2n}{z}\frac{g'(z)}{g(z)} + \frac{g''(z)}{g(z)}.$$

So the residue equals to $-2n\frac{g'(0)}{g(0)}$. □

(3)

Proof.

$$\frac{f''(z)}{f'(z)} = \frac{n(n+1)z^{-n-2}g(z) - 2nz^{-n-1}g'(z) + z^{-n}g''(z)}{-nz^{-n-1}g(z) + z^{-n}g'(z)} = \frac{1}{z} \cdot \frac{n(n+1)g(z) - 2nzg'(z) + z^2g''(z)}{-ng(z) + zg'(z)}.$$

Note the second term in the above product is analytic near 0, so the residue is equal to

$$\frac{n(n+1)g(z) - 2nzg'(z) + z^2g''(z)}{-ng(z) + zg'(z)}|_{z=0} = -(n+1).$$

□

(4)

Proof. $(n+1)f'(z) + zf''(z) = -(n+1)nz^{-n-1}g(z) + (n+1)z^{-n}g'(z) + n(n+1)z^{-n-1}g(z) - 2nz^{-n}g'(z) + z^{-n+1}g''(z) = -(n-1)z^{-n}g'(z) + z^{-n+1}g''(z)$. So

$$\frac{(n+1)f'(z) + zf''(z)}{f(z)} = \frac{-(n-1)z^{-n}g'(z) + z^{-n+1}g''(z)}{z^{-n}g(z)} = -(n-1)\frac{g'(z)}{g(z)} + \frac{zg''(z)}{g(z)}.$$

So the residue is equal to 0. □

7.4.

Proof.

function	given conditions	type of singularity	residue
$\frac{g(z)}{f(z)}$	z_0 are zeros of $g(z)$, $f(z)$ and has the same order	removable	0
$\frac{g(z)}{f(z)}$	$g(z_0) \neq 0$, $f(z_0) = 0$, $f'(z_0) \neq 0$	pole of order 1	$\frac{g(z_0)}{f'(z_0)}$
$\frac{g(z)}{f(z)}$	z_0 is zero of $g(z)$ of order m and zero of $f(z)$ of order m+1	pole of order 1	$\frac{(m+1)g^{(m)}(z_0)}{f^{(m+1)}(z_0)}$
$\frac{g(z)}{f(z)}$	$g(z_0) \neq 0$, $f(z_0) = f'(z_0) = 0$, $f''(z_0) \neq 0$	pole of order 2	
$\frac{g(z)}{(z-z_0)^2}$	$g(z_0) \neq 0$	pole of order 2	$g'(z_0)$
$\frac{g(z)}{f(z)}$	z_0 is zero of $f(z)$ of order m and $g(z_0) \neq 0$	pole of order m	
$\frac{g(z)}{f(z)}$	z_0 is zero of $g(z)$ of order m and zero of $f(z)$ of order m+n	pole of order n	

□

7.5.

Proof. $f^2(z) = c_0^2 + \frac{2c_0c_1}{z} + \frac{1}{z^2}P(z)$, where $P(z)$ is a power series of $\frac{1}{z}$. By formula (7.12), $\text{Res}(f, \infty) = -2c_0c_1$. □

7.6.

Proof. Suppose the singularities of f are $a_1, \cdots, a_n$. Let R be large enough so that all the finite singularities of f fall in the disc $|z| < R$. Then Cauchy's theorem implies

$$\frac{1}{2\pi i}\int_{|z|=R} f(z)dz - \sum_{i=1,|a_i|<\infty}^{n} \text{Res}(f, a_i) = 0,$$

i.e. $\sum_{i=1}^{n} \text{Res}(f, a_i) + \text{Res}(f, \infty) = 0$. So the sum of residues of f on $\bar{\mathbb{C}}$ is 0. □

7.7.

Proof. Let $\theta_k = \arg\beta_k$ $(k = 1, 2, \cdots, m)$ and ρ be a positive number that is sufficiently small. Define $C_\rho = \{z : |z| = 1, |\arg z - \theta_k| \geq \rho, 1 \leq k \leq m\}$. Define γ_k as the arc that starts from $e^{i(\theta_k-\rho)}$, ends at $e^{i(\theta_k+\rho)}$, has $\beta_k = e^{i\theta_k}$ as the center, and dents toward origin. Then by Residue Theorem, for $\rho > 0$ sufficiently small, we have

$$\int_{C_\rho \cup \gamma_1 \cup \cdots \cup \gamma_m} \frac{f(z)}{iz} dz = 2\pi \sum_{|z|<1} res\left\{\frac{f(z)}{z}\right\}.$$

Then for each $k \in \{1, 2, \cdots, m\}$, we have

$$\int_{-\gamma_k} \frac{f(z)}{z} dz = \int_{\phi_1(\rho)}^{\phi_2(\rho)} \frac{f(\beta_k + \rho e^{i\alpha})}{\beta_k + \rho e^{i\alpha}} \rho e^{i\alpha} i d\alpha,$$

where $\phi_2(\rho) - \phi_1(\rho) \to \pi$ as $\rho \to 0$. Since β_k is a pole of order 1, in a neighborhood of β_k, we can write $f(z)$ as $\frac{g(z)}{z-\beta_k}$ where $g(z)$ is analytic near β_k. Then (note $res\left\{\frac{f(z)}{z}, \beta_k\right\} = \frac{g(\beta_k)}{\beta_k}$)

$$\int_{-\gamma_k} \frac{f(z)}{z} dz = \int_{\phi_1(\rho)}^{\phi_2(\rho)} \frac{g(\beta_k + \rho e^{i\alpha})}{(\beta_k + \rho e^{i\alpha})\rho e^{i\alpha}} \rho e^{i\alpha} i d\alpha \to \pi i \frac{g(\beta_k)}{\beta_k} = i\pi res\left\{\frac{f(z)}{z}, \beta_k\right\}, \text{ as } \rho \to 0.$$

Therefore

$$\begin{aligned}\int_0^{2\pi} R(\sin\theta, \cos\theta) d\theta &= \lim_{\rho\to 0} \int_{C_\rho} \frac{f(z)}{iz} dz = 2\pi \sum_{|z|<1} res\left\{\frac{f(z)}{z}\right\} + \lim_{\rho\to 0} \sum_{k=1}^m \int_{-\gamma_k} \frac{f(z)}{iz} dz \\ &= 2\pi \sum_{|z|<1} res\left\{\frac{f(z)}{z}\right\} + \pi \sum_{k=1}^m res\left\{\frac{f(z)}{z}, \beta_k\right\}.\end{aligned}$$

Remark 5. *The above result and the trick of indenting the contour can be found in Whittaker and Watson [11], §6.23, page 117.*

□

7.8.

Proof. We compute a more general integral $\int_0^\infty \frac{dx}{1+x^p}$ where $p \in (1, \infty)$. Choose two positive numbers r and R such that $0 < r < 1 < R$. Let $\gamma_1 = \{z : r \leq |z| \leq R, \arg z = 0\}$, $\gamma_2 = \{z : r \leq |z| \leq R, \arg z = 2\pi\}$, $\gamma_R = \{z : |z| = R, 0 < \arg z < 2\pi\}$ and $\gamma_r = \{z : |z| = r, 0 < \arg z < 2\pi\}$. Define $f(z) = \frac{z^{1/p}}{p(z+1)z}$ where $z^{1/p} = e^{\frac{\log z}{p}}$ is defined on $\mathbb{C} \setminus [0, \infty)$. Note by substituting $y^{\frac{1}{p}}$ for x, we get

$$\int_0^\infty \frac{dx}{1+x^p} = \int_0^\infty \frac{y^{\frac{1}{p}} dy}{p(y+1)y}.$$

By Residue Theorem,

$$\int_{\gamma_1+\gamma_R-\gamma_2-\gamma_r} f(z) dz = 2\pi \text{Res}(f, -1) = 2\pi i \cdot \frac{(-1)^{\frac{1}{p}}}{-p} = -\frac{2\pi i}{p} e^{\frac{\log e^{\pi i}}{p}} = -2i\alpha e^{\alpha i},$$

where $\alpha = \frac{\pi}{p}$. We have the estimates

$$\left|\int_{\gamma_R} f(z) dz\right| = \left|\int_0^{2\pi} \frac{e^{\frac{1}{p}\log(Re^{i\theta})}}{p(Re^{i\theta}+1)Re^{i\theta}} Re^{i\theta} \cdot i d\theta\right| \leq \frac{2\pi R^{\frac{1}{p}}}{p(R-1)} \to 0$$

as $R \to \infty$,

$$\left|\int_{\gamma_r} f(z) dz\right| = \left|\int_0^{2\pi} \frac{e^{\frac{1}{p}\log(re^{i\theta})}}{p(re^{i\theta}+1)re^{i\theta}} re^{i\theta} \cdot i d\theta\right| \leq \frac{2\pi r^{\frac{1}{p}}}{p(1-r)} \to 0$$

as $r \to 0$, and

$$\int_{\gamma_2} f(z)dz = -\int_r^R \frac{(xe^{2\pi i})^{\frac{1}{p}}dx}{p(x+1)x} = -\int_r^R \frac{x^{\frac{1}{p}}dx}{p(x+1)x} \cdot e^{2\alpha i}.$$

Therefore by letting $r \to 0$ and $R \to \infty$, we have

$$\int_0^\infty \frac{dx}{1+x^p} = \int_0^\infty \frac{x^{\frac{1}{p}}dx}{p(x+1)x} = \frac{-2i\alpha e^{\alpha i}}{1-e^{2\alpha i}} = \frac{-2i\alpha e^{\alpha i}}{-2\sin\alpha e^{(\frac{\pi}{2}+\alpha)i}} = \frac{\alpha}{\sin\alpha} = \frac{\pi}{p}\csc\left(\frac{\pi}{p}\right).$$

□

7.9.

Proof. This is a special case of exercise 7.10. Answer: $\frac{\pi^4}{90}$ (verified via Mathematica: **Sum[1/n^ 4, n, 1, Infinity]).** □

7.10.

Proof. Let C_N be the contour used in Lemma 7.2: $C_N = [N+\frac{1}{2}, (N+\frac{1}{2})i, -(N+\frac{1}{2}), -(N+\frac{1}{2})i, N+\frac{1}{2}]$. Then by Residue Theorem,

$$\oint_{C_N} \frac{\pi\cot\pi z}{z^{2k}}dz = 2\pi i \sum_{n=-N}^{N} Res\left(\frac{\pi\cot\pi z}{z^{2k}}, n\right).$$

$z=0$ is a pole of order 1 for $\cot z$. So we can assume the Laurent series of $\cot z$ near 0 is $\sum_{n=0}^\infty b_{2n-1}z^{2n-1}$. Then

$$\frac{\pi\cot\pi z}{z^{2k}} = \sum_{n=0}^\infty \frac{\pi\cdot b_{2n-1}(\pi z)^{2n-1}}{z^{2k}} = \sum_{n=0}^\infty \frac{b_{2n-1}\pi^{2n}}{z^{2(n-k)+1}}$$

Therefore, for $\rho > 0$ sufficiently small,

$$Res\left(\frac{\pi\cot\pi z}{z^{2k}}, 0\right) = \frac{1}{2\pi i}\int_{|z|=\rho} \frac{\pi\cot\pi z}{z^{2k}}dz = b_{2k-1}\pi^{2k}.$$

For $n \neq 0$, take $\rho > 0$ sufficiently large, we have

$$Res\left(\frac{\pi\cot\pi z}{z^{2k}}, n\right) = \frac{1}{2\pi i}\int_{|z-n|=\rho} \frac{(-1)^n\cos\pi z\cdot\pi(z-n)}{z^{2k}\sin\pi(z-n)}\cdot\frac{1}{z-n}dz = \frac{1}{n^{2k}}.$$

Combined, we can conclude

$$\oint_{C_N} \frac{\pi\cot\pi z}{z^{2k}}dz = 2\pi i\left(b_{2k-1}\pi^{2k} + 2\sum_{n=1}^N \frac{1}{n^{2k}}\right).$$

By Lemma 7.2, $\lim_{N\to\infty}\oint_{C_N}\frac{\pi\cot\pi z}{z^{2k}}dz = 0$. So $\sum_{n=1}^\infty \frac{1}{n^{2k}} = -\frac{b_{2k-1}}{2}\pi^{2k}$. From formula (5.25), we have $b_{-1} = 1$, $b_1 = -\frac{1}{3}$, $b_3 = -\frac{1}{45}$, $b_5 = -\frac{2}{945}$,etc. In particular, for $k=2$, $b_{2k-1} = b_3 = -\frac{1}{45}$. So $\sum_{n=1}^\infty \frac{1}{n^4} = \frac{\pi^4}{90}$, which gives the answer to exercise 7.9. □

7.11.

Proof. We first deduce the Laurent series of $\frac{1}{\sin z}$ near 0. 0 is a pole of order 1 for $\frac{1}{\sin z}$. Note $\sin z$ is an odd function, we can assume its Laurent series near 0 is $\sum_{l=0}^\infty b_{2l-1}z^{2l-1}$. Then

$$1 = \sin z\cdot\sum_{n=0}^\infty b_{2n-1}z^{2n-1} = \sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)!}z^{2k+1}\cdot\sum_{l=0}^\infty b_{2l-1}z^{2l-1} = \sum_{n=0}^\infty\left[\sum_{l=0}^n b_{2l-1}\frac{(-1)^{(n-l)}}{(2(n-l)+1)!}\right]z^{2n}.$$

So $b_{-1}=1$ and $\sum_{l=0}^{n} b_{2l-1}\frac{(-1)^{(n-l)}}{(2(n-l)+1)!}=0$ for $n\geq 1$. Using this, we can easily get the Laurent series of $\frac{1}{\sin z}$ (verified by Mathematica: **Series**[**z**/**Sin**[**z**], **{z, 0, 10}**])

$$\frac{1}{\sin z}=\frac{1}{z}+\frac{z}{6}+\frac{7z^3}{360}+\frac{31z^5}{15120}+\frac{127z^7}{604800}+\frac{73z^9}{3421440}+o(z^{10}).$$

Let C_N be the contour used in Lemma 7.2: $C_N=[N+\frac{1}{2},(N+\frac{1}{2})i,-(N+\frac{1}{2}),-(N+\frac{1}{2})i,N+\frac{1}{2}]$. Then by Residue Theorem,

$$\oint_{C_N}\frac{\pi}{z^2\sin\pi z}dz=2\pi i\sum_{n=-N}^{N}Res\left(\frac{\pi}{z^2\sin\pi z},n\right).$$

From the Laurent series of $\frac{1}{\sin\pi z}$, we can deduce

$$Res\left(\frac{\pi}{z^2\sin\pi z},0\right)=\frac{\pi^2}{6}.$$

For $n\neq 0$, we can find $\rho>0$ sufficiently small, so that

$$Res\left(\frac{\pi}{z^2\sin\pi z},n\right)=\frac{1}{2\pi i}\int_{|z-n|=\rho}\frac{(-1)^n\pi(z-n)}{z^2\sin\pi(z-n)}\cdot\frac{1}{z-n}dz=\frac{(-1)^n}{n^2}.$$

Therefore $\oint_{C_N}\frac{\pi}{z^2\sin\pi z}dz=2\pi i\left[\frac{\pi^2}{6}-2\sum_{n=1}^{N}\frac{(-1)^{n-1}}{n^2}\right]$. Suppose $\lim_{N\to\infty}\oint_{C_N}\frac{\pi}{z^2\sin\pi z}dz=0$, then we can conclude $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}=\frac{\pi^2}{12}$ (verified by Mathematica: **Sum**[**(-1)^ (n - 1)/n^ 2, n, 1, Infinity**]). □

7.2 Exercises at the end of chapter

1. (1)

Proof. $Res\left(\frac{e^{z^2}}{z-1},1\right)=e^{z^2}|_{z=1}=e.$ □

(2)

Proof. $Res\left(\frac{e^{z^2}}{(z-1)^2},1\right)=\frac{d}{dz}e^{z^2}|_{z=1}=2e.$ □

(3)

Proof. $1-\cos z=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n)!}z^{2n}$. So 0 is a pole of order 2 for $\left(\frac{z}{1-\cos z}\right)^2$. For sufficiently small $\rho>0$, we have

$$Res\left(\left(\frac{z}{1-\cos z}\right)^2,0\right)=\frac{1}{2\pi i}\int_{|z|=\rho}\frac{z^4}{(1-\cos z)^2}\cdot\frac{dz}{z^2}=\frac{d}{dz}\left[\frac{z^4}{(1-\cos z)^2}\right]\Bigg|_{z=0}$$

By repeatedly applying l'Hospitale's rule for analytic functions (see exercise 5.1, 5.2 in the text, page 62) and the fact $\lim_{z\to 0}\frac{\sin z}{z}=1$, we can conclude the residue is equal to 0. □

(4)

Proof. By repeatedly using l'Hospitale's rule for analytic functions and the fact $\lim_{z\to 0}\frac{\sin z}{z}=1$, we have

$$Res\left(\frac{1}{z^2\sin z},0\right)=Res\left(\frac{1}{z^3}\cdot\frac{z}{\sin z},0\right)=\frac{d^2}{dz^2}\left(\frac{z}{\sin z}\right)\Bigg|_{z=0}=\frac{1}{6}.$$

□

(5)

Proof.

$$Res\left(\frac{e^z}{z^2-1},1\right) = Res\left(\frac{e^z}{(z-1)^2(z+1)^2},1\right) = \frac{d}{dz}\left.\frac{e^z}{(z+1)^2}\right|_{z=1} = 0.$$

□

(6)

Proof. Let $z_n = -\left(\frac{2n+1}{2}\pi\right)^2$. If we take the convention that $\sqrt{-1}=i$, we have $\sqrt{z_n} = (n\pi+\frac{\pi}{2})i$. It's easy to see $\cosh\sqrt{z_n} = \cosh[(n\pi+\frac{\pi}{2})i] = \cos(n\pi+\frac{\pi}{2}) = 0$, and

$$\lim_{z\to z_n}(\cosh\sqrt{z})' = \lim_{z\to z_n}\frac{i\sinh\sqrt{z}}{2\sqrt{z}} = \frac{\sin(n\pi+\frac{\pi}{2})}{(2n+1)\pi} = \frac{(-1)^n}{(2n+1)\pi}.$$

So z_n is a pole of order 1 for $\frac{1}{\cosh\sqrt{z}}$ and

$$Res\left(\frac{1}{\cosh\sqrt{z}},z_n\right) = Res\left(\frac{z-z_n}{\cosh\sqrt{z}}\cdot\frac{1}{z-z_n},z_n\right) = \lim_{z\to z_n}\frac{z-z_n}{\cosh\sqrt{z}} = \frac{1}{(\cosh\sqrt{z})'|_{z=z_n}} = (-1)^n(2n+1)\pi.$$

□

2. (1)

Proof. Let $f(z) = \frac{1}{z^3(1-z)(1+z)}$. Then

$$\text{Res}(f,0) = \frac{1}{2!}\frac{d^2}{dz^2}\left.\frac{1}{1-z^2}\right|_{z=0} = 1,$$

$\text{Res}(f,1) = -\frac{1}{2}$, and $\text{Res}(f,-1) = -\frac{1}{2}$. ∞ is a removable singularity. □

(2)

Proof. ∞ is a removable singularity. Let $f(z) = \frac{1}{(z^2+1)^{m+1}} = \frac{1}{(z+i)^{m+1}}\cdot\frac{1}{(z-i)^{m+1}}$. Then

$$\text{Res}(f,i) = \frac{1}{m!}\frac{d^m}{dz^m}\left.\left[\frac{1}{(z+i)^{m+1}}\right]\right|_{z=i} = \frac{1}{m!}[-(m+1)][-(m+2)]\cdots(-2m)(z+i)^{-2m-1}\Big|_{z=i} = \frac{-i}{2^{2m+1}}\frac{(2m)!}{(m!)^2}.$$

Similarly, we have $\text{Res}(f,-i) = \frac{i}{2^{2m+1}}\frac{(2m)!}{(m!)^2}$. □

(3)

Proof. $1-\cos z = 2\sin^2\frac{z}{2}$. So $z_n = 2n\pi$ $(n\in\mathbb{Z})$ is a pole of order 2 for the function $f(z) = \frac{z}{1-\cos z}$. Note for each n,

$$f(z) = \frac{z}{2\sin^2\frac{z}{2}} = \frac{z}{2\left(\frac{z-2n\pi}{2}\right)^2}\cdot\frac{\left(\frac{z-2n\pi}{2}\right)^2}{\sin^2\frac{z-2n\pi}{2}} = \frac{1}{(z-2n\pi)^2}\cdot\frac{2z\left(\frac{z-2n\pi}{2}\right)^2}{\sin^2\frac{z-2n\pi}{2}}.$$

Define $h(w) = \frac{w}{\sin w}$. Since the Laurent series of $\frac{1}{\sin w}$ near 0 is

$$\frac{1}{\sin z} = \frac{1}{z}+\frac{z}{6}+\frac{7z^3}{360}+\frac{31z^5}{15120}+\frac{127z^7}{604800}+\frac{73z^9}{3421440}+o(z^{10}),$$

we have the Luarent series of $h(w)$ near 0:

$$h(w) = \frac{w}{\sin w} = 1+\frac{w^2}{6}+\frac{7w^4}{360}+\frac{31w^6}{15120}+\frac{127w^8}{604800}+\frac{73w^{10}}{3421440}+o(w^{11}).$$

Therefore, $h(0) = 1$ and $h'(0) = 0$. Moreover,

$$\begin{aligned} \text{Res}(f, 2n\pi) &= Res\left(\frac{1}{(z-2n\pi)^2}\cdot 2zh^2\left(\frac{z-2n\pi}{2}\right), 2n\pi\right) \\ &= \lim_{z\to 2n\pi}\frac{d}{dz}\left[2zh^2\left(\frac{z}{2}-n\pi\right)\right] \\ &= \lim_{z\to 2n\pi}\left[2h^2\left(\frac{z}{2}-n\pi\right)+2z\cdot 2h\left(\frac{\pi}{2}-n\pi\right)h'\left(\frac{z}{2}-n\pi\right)\cdot\frac{1}{2}\right] \\ &= 2. \end{aligned}$$

□

(4)

Proof. Let $f(z) = \frac{\sqrt{z}}{\sinh\sqrt{z}}$. Then $\sinh\sqrt{z} = 0$ if and only if $z = -(n\pi)^2$ ($n \in \mathbb{Z}$). Let $z_n = -(n\pi)^2$. Then $(z_n)_{n\in\mathbb{Z}}$ are singularities of $f(z)$. For $n = 0$, $z_0 = 0$ is a removable singularity since

$$\lim_{z\to 0}\frac{\sqrt{z}}{\sinh\sqrt{z}} = \frac{1}{\cosh 0} = 1.$$

Therefore $\text{Res}(f, 0) = 0$. For $n \neq 0$, we have (suppose $\rho > 0$ is sufficiently small)

$$\text{Res}(f, z_n) = \frac{1}{2\pi i}\int_{|z-z_n|=\rho}\frac{1}{z-z_n}\cdot\frac{\sqrt{z}(\sqrt{z}-\sqrt{z_n})(\sqrt{z}+\sqrt{z_n})}{\sinh\sqrt{z}-\sinh\sqrt{z_n}}dz = \frac{\sqrt{z_n}(\sqrt{z_n}+\sqrt{z_n})}{\cosh\sqrt{z_n}} = 2z_n = -2(n\pi)^2.$$

Remark 6. *This solution has a different result from that of the textbook's solution. Check.*

□

(5)

Proof. Both 0 and ∞ are essential singularities of $f(z) = \exp\left[\frac{1}{2}\left(z - \frac{1}{z}\right)\right]$, as can be seen by letting $z \to \infty$ and $z \to 0$ along positive and negative real axis.

To find the residue of the function at 0, we note

$$\exp\left[\frac{1}{2}\left(z-\frac{1}{z}\right)\right] = \sum_{n=0}^{\infty}\frac{1}{2^n n!}\left(z-\frac{1}{z}\right)^n.$$

For each n,

$$\left(z-\frac{1}{z}\right)^n = \sum_{k=0}^{n}\binom{k}{n}z^k(-z^{-1})^{n-k}.$$

So the expansion of $\left(z - \frac{1}{z}\right)^n$ contains z^{-1} term if and only if n is an odd number, and in this case, the coefficient of z^{-1} is $(-1)^{\frac{n+1}{2}}\binom{\frac{n-1}{2}}{n}$. So in the expansion of $\left(z - \frac{1}{z}\right)^n$, the coefficient of z^{-1} is

$$\sum_{m=0}^{\infty}(-1)^{m+1}\frac{1}{2^{2m+1}(2m+1)!}\binom{m}{2m+1} = \sum_{m=0}^{\infty}\frac{(-1)^{m+1}\left(\frac{1}{2}\right)^{2m+1}}{m!(m+1)!} = -J_1(1),$$

where $J_\alpha(z)$ is Bessel function of the first kind:

$$J_\alpha(z) = \sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(m+\alpha+1)}\left(\frac{z}{2}\right)^{2m+\alpha}.$$

So $\text{Res}(f, 0) = -J_1(1)$. Since the above expansion is valid in $\mathbb{C}\setminus\{0\}$, by remark on page 87 (formula (7.12)), we conclude $\text{Res}(f, \infty) = -\text{Res}(f, 0) = J_1(1)$.

Remark 7. *The results in the above solution have signs opposite to those of the textbook's solution. Check.* □

(6)

Proof. ∞ is a removable singularity and 0 is a pole. Using the Taylor series of cosine function near 0, we have

$$\cos\frac{1}{\sqrt{z}} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\frac{1}{z^n}.$$

So the residue is equal to the coefficient of $\frac{1}{z}$, which is $-\frac{1}{2}$. □

(7)

Proof. Let $f(z) = \frac{1}{(z-1)\ln z}$. Then 1 is a pole of $f(z)$ and ∞ is a removable singularity. When $\ln 1 = 2n\pi i$ ($n \in \mathbb{Z}\setminus\{0\}$), for $\rho > 0$ sufficiently small,

$$\mathrm{Res}(f,1) = \frac{1}{2\pi i}\int_{|z-1|=\rho} f(z)dz = \frac{1}{\ln 1} = \frac{1}{2n\pi i}.$$

When $\ln 1 = 0$, $\frac{z-1}{\ln z}$ is analytic near 0. So for $\rho > 0$ sufficiently small, by applying l'Hospitale's rule, we have

$$\mathrm{Res}(f,1) = \frac{1}{2\pi i}\int_{|z-1|=\rho}\frac{1}{(z-1)^2}\cdot\frac{z-1}{\ln z}dz = \frac{d}{dz}\left[\frac{z-1}{\ln z}\right]\Big|_{z=1} = \frac{1}{2}.$$

□

(8)

Proof. Let $f(z) = \frac{1}{z}\left[1 + \frac{1}{z+1} + \frac{1}{(z+1)^2} + \cdots + \frac{1}{(z+1)^n}\right]$. Then 0 and -1 are poles of f, while ∞ is a removable singularity of f. For $\rho > 0$ sufficiently small,

$$\begin{aligned}\mathrm{Res}(f,0) &= \frac{1}{2\pi i}\int_{|z|=\rho}\frac{1}{z}\left[1 + \frac{1}{z+1} + \frac{1}{(z+1)^2} + \cdots + \frac{1}{(z+1)^n}\right]dz \\ &= \left[1 + \frac{1}{z+1} + \frac{1}{(z+1)^2} + \cdots + \frac{1}{(z+1)^n}\right]\Big|_{z=0} \\ &= n+1.\end{aligned}$$

Since $f(z)$ can be written as

$$\frac{1}{z}\frac{1-\frac{1}{(z+1)^{n+1}}}{1-\frac{1}{z+1}} = \frac{1}{z^2}\left[(z+1) - \frac{1}{(z+1)^n}\right],$$

we have

$$\mathrm{Res}(f,-1) = \frac{1}{2\pi i}\int_{|z+1|=\rho}\frac{-1}{z^2(z+1)^n}dz = -\frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}z^{-2}\Big|_{z=-1} = -\frac{(-2)\cdots(-n)z^{-(n+1)}}{(n-1)!}\Big|_{z=-1} = -n.$$

□

3. (1)

Proof. Note $\frac{1}{z}$ is analytic on $\mathbb{C}\setminus\{0\}$, so $\mathrm{Res}(\frac{1}{z},\infty) = -\mathrm{Res}(\frac{1}{z},0) = -1$. □

(2)

Proof. Note $\frac{\cos z}{z}$ is analytic on $\mathbb{C}\setminus\{0\}$, so $\mathrm{Res}(\frac{\cos z}{z},\infty) = -\mathrm{Res}(\frac{\cos z}{z},0) = -1$. □

(3)

Proof. Since $\cos(2n\pi + \frac{\pi}{2}) = 0$ ($n \in \mathbb{Z}$), ∞ is not an isolated singularity. $\square$

(4)

Proof. Since $(z^2+1)e^z$ is analytic on $\mathbb{C}$, $\mathrm{Res}((z^2+1)e^z, \infty) = 0$. $\square$

(5)

Proof. $e^{-\frac{1}{z^2}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{2n} n!}$, which has no z^{-1} term. So $\mathrm{Res}(e^{-\frac{1}{z^2}}, \infty) = 0$. $\square$

(6)

Proof. Recall $\mathrm{Res}(\sqrt{(z-1)(z-2)}, \infty)$ is equal to the coefficient of the term z in the power series expansion of $-\sqrt{\left(\frac{1}{z}-1\right)\left(\frac{1}{z}-2\right)}$ near 0. By generalized Newton's formula, we have

$$\sqrt{(1-z)(1-2z)} = 1 - \frac{3z}{2} - \frac{z^2}{8} - \frac{3z^3}{16} - \cdots$$

Depending on the branch we choose, we have $\sqrt{z^2} = \pm z$. So the power series expansion of $-\sqrt{\left(\frac{1}{z}-1\right)\left(\frac{1}{z}-2\right)}$ near 0 is

$$\mp\left(\frac{1}{z} - \frac{3}{2} - \frac{z}{8} - \frac{3z^2}{16} - \cdots\right).$$

Therefore, $\mathrm{Res}(\sqrt{(z-1)(z-2)}, \infty) = \pm\frac{1}{8}$. $\square$

4. (1)

Proof. The equation $z^4+1=0$ has four roots: $z_1 = e^{\frac{\pi}{4}i}$, $z_2 = e^{-\frac{\pi}{4}i}$, $z_3 = e^{\frac{3\pi}{4}i}$, and $z_4 = e^{\frac{5\pi}{4}i}$. The intersection points of $|z-1|=1$ and $|z|=1$ are $e^{\pm\frac{\pi}{3}i}$. So only z_1 and z_2 fall within the disc $|z-1|<1$. By Residue Theorem, we have

$$\begin{aligned}
\oint_{|z-1|=1} \frac{dz}{1+z^4} &= 2\pi i \left[Res\left(\frac{1}{1+z^4}, e^{\frac{\pi}{4}i}\right) + Res\left(\frac{1}{1+z^4}, e^{-\frac{\pi}{4}i}\right)\right] \\
&= 2\pi i \left(\lim_{z\to z_1} \frac{z-z_1}{z^4+1} + \lim_{z\to z_2} \frac{z-z_2}{z^4+1}\right) \\
&= 2\pi i \cdot \left(\frac{1}{4z_1^3} + \frac{1}{4z_2^3}\right) \\
&= 2\pi i \cdot \frac{z_1+z_2}{-4} \\
&= -\frac{\pi i}{\sqrt{2}}.
\end{aligned}$$

$\square$

(2)

Proof. Using the same notation as in part (1), we first show all the roots of $z^4+1=0$ fall within the circle $|z-1|=2$. Indeed, for any $\theta \in [0, 2\pi)$, $|e^{i\theta}-1| = 2|\sin\frac{\theta}{2}| \le 2$, where the equality holds if and only if $\theta = \pi$. So z_i's ($i = 1,2,3,4,$) all fall within the circle $|z-1|=2$. Then by Residue Theorem and an argument similar to part (1), we have

$$\oint_{|z-1|=2} \frac{dz}{1+z^4} = 2\pi i \cdot \frac{z_1+z_2+z_3+z_4}{-4} = 0.$$

$\square$

(3)

Proof. By Residue Theorem,

$$\oint_{|z-1|=1} \frac{1}{z^2-1} \sin \frac{\pi z}{4} dz = 2\pi i \cdot \frac{\sin \frac{\pi}{4}}{1+1} = \frac{2i}{\sqrt{2}}.$$

□

(4)

Proof. By Residue Theorem,

$$\oint_{|z|=3} \frac{1}{z^2-1} \sin \frac{\pi z}{4} dz = 2\pi i \left(\frac{\sin \frac{\pi}{4}}{1+1} + \frac{\sin\left(-\frac{\pi}{4}\right)}{-1-1} \right) = \sqrt{2}\pi i.$$

□

(5)

Proof. The singularities that fall within the circle $|z| = n$ are $k + \frac{1}{2}$ with $k = -n, -n+1, \cdots, n-1$. By Residue Theorem, we have

$$\oint_{|z|=n} \tan \pi z dz = 2\pi i \sum_{k=-n}^{n-1} \lim_{z \to k+\frac{1}{2}} \frac{[z-(k+\frac{1}{2})]\sin(\pi z)}{\cos(\pi z)} = 2\pi i \sum_{k=-n}^{n-1} \frac{\sin\left(k\pi + \frac{1}{2}\pi\right)}{-\pi \sin\left(k\pi + \frac{1}{2}\pi\right)} = -4ni.$$

□

(6)

Proof. Let z_n $(1 \le n \le 10)$ be the n-th root of the equation $z^{10} = 2$. For example, we can let $z_n = 2^{\frac{1}{10}} e^{\frac{2n\pi i}{10}}$. Then similar to our solution of part (1), Residue Theorem gives

$$\oint_{|z|=2} \frac{dz}{z^3(z^{10}-2)} = 2\pi i \left[\frac{1}{2!} \frac{d^2}{dz^2}(z^{10}-2)^{-2}\Big|_{z=0} + \sum_{n=1}^{10} \frac{1}{z_n^3} \frac{1}{10 z_n^9} \right] = \frac{\pi i}{10} \sum_{i=1}^{10} \frac{1}{z_i^2} = \frac{\pi i}{10 \cdot 2^{\frac{1}{5}}} \sum_{i=1}^{10} e^{-\frac{2n\pi i}{5}}.$$

Note $\sum_{i=1}^{10} e^{-\frac{2n\pi i}{5}} = e^{-\frac{2\pi i}{5}} \frac{1-(e^{-\frac{2\pi i}{5}})^{10}}{1-e^{-\frac{2\pi i}{5}}} = 0$. So the integral is evaluated to 0. □

(7)

Proof. By Residue Theorem,

$$\oint_{|z|=1} \frac{e^z}{z^3} dz = 2\pi i \cdot \frac{1}{2!} \frac{d^2}{dz^2} e^z \Big|_{z=0} = \pi i.$$

□

(8)

Proof. We note $e^{2\pi i z^3} - 1 = 0$ if and only if for some $k \in \mathbb{Z}$, $z^3 = k$. Since $n < R^3 < n+1$, a number z_* is a root of $e^{2\pi i z^3} - 1 = 0$ within the circle $|z| = R$ if and only if $z_*^3 = k$ for some $k \in [-n, n]$. Suppose those roots are z_j. Then by Residue Theorem, we have (assume $\rho > 0$ is sufficiently small)

$$\oint_{|z|=R} \frac{z^2}{e^{2\pi i z^3}-1} dz = 2\pi i \sum_{z_j \neq 0} \frac{z_j^2}{2\pi i \cdot 3 z_j^2 e^{2\pi i z_j^3}} + \oint_{|z|=\rho} \frac{z^2}{2\pi i z^3 + \frac{1}{2!}(2\pi i z^3)^2 + \cdots} dz = \frac{1}{3} \sum_{j \neq 0, |j| \le n} 1 + 1 = \frac{2n}{3} + 1.$$

Remark 8. *Note our result is different from the textbook's solution.*

□

5. (1)

Proof.

$$\begin{aligned}\int_0^{2\pi} \cos^{2n}\theta d\theta &= \int_{|z|=1}\left(\frac{z+z^{-1}}{2}\right)^{2n}\frac{dz}{iz} = \frac{2\pi}{2^{2n}(2n)!}\frac{(2n)!}{2\pi i}\int_{|z|=1}\frac{(z^2+1)^{2n}}{z^{2n+1}}dz \\ &= \frac{2\pi}{2^{2n}(2n)!}\frac{d^{2n}}{dz^{2n}}(z^2+1)^{2n}\Big|_{z=0} = \frac{2\pi}{2^{2n}(2n)!}\frac{d^{2n}}{dz^{2n}}\binom{n}{2n}z^{2n}\Big|_{z=0} \\ &= \frac{\pi}{2^{2n-1}}\frac{(2n)!}{(n!)^2}.\end{aligned}$$

□

(2)

Proof.

$$\int_0^{2\pi}\frac{dx}{(a+b\cos x)^2} = \int_{|z|=1}\frac{1}{\left(a+b\frac{z+z^{-1}}{2}\right)^2}\frac{dz}{iz} = \int_{|z|=1}\frac{4z}{(bz^2+2az+b)^2}\frac{dz}{i}.$$

The equation $bz^2+2az+b=0$ has two solutions: $z_1=\frac{-a+\sqrt{a^2-b^2}}{b}$ and $z_2=\frac{-a-\sqrt{a^2-b^2}}{b}$. Clearly $|z_2|>1$ and $|z_1|<1$. So by Residue Theorem, we have ($f(z):=\frac{4z}{(bz^2+2az+b)^2 i}$ and $\rho>0$ is sufficiently small)

$$\begin{aligned}\int_0^{2\pi}\frac{dx}{(a+b\cos)^2} &= 2\pi i \mathrm{Res}(f,z_1) \\ &= \int_{|z-z_1|=\rho}\frac{4z}{b^2(z-z_1)^2(z-z_2)^2}\frac{dz}{i} \\ &= \frac{4}{b^2 i}\cdot 2\pi i\,\frac{d}{dz}\frac{z}{(z-z_2)^2}\Big|_{z=z_1} \\ &= \frac{8\pi}{b^2}\frac{z_1+z_2}{(z_2-z_1)^3} \\ &= \frac{2a\pi}{(a^2-b^2)^{3/2}}.\end{aligned}$$

□

(3)

Proof. We note $\sin^2\theta=\frac{1}{2}[1-\cos(2\theta)]$. Using the substitution rule $\theta=\frac{1}{2}\alpha$, we have

$$\int_0^{\pi}\frac{d\theta}{1+\sin^2\theta} = \int_0^{2\pi}\frac{d\alpha}{3-\cos\alpha} = 2i\int_{|z|=1}\frac{dz}{z^2-6z+1}.$$

The equation $z^2-6z+1=0$ has two roots: $z_1=3+2\sqrt{2}$ and $z_2=3-2\sqrt{2}$. It's clear that $|z_1|>1$ and $|z_2|<1$. By Residue Theorem, we have

$$\int_0^{\pi}\frac{d\theta}{1+\sin^2\theta} = 2i\cdot\frac{2\pi i}{2\pi i}\int_{|z|=1}\frac{dz}{(z-z_1)(z-z_2)} = -4\pi\cdot\frac{1}{z_2-z_1} = \frac{\pi}{\sqrt{2}}.$$

□

(4)

Proof. Similar to problem (3), we have the following argument:

$$\begin{aligned}\int_0^{\pi}\frac{d\theta}{(1+\sin^2\theta)^2} &= 2\int_0^{2\pi}\frac{d\alpha}{(3-\cos\alpha)^2}=\frac{8}{i}\int_{|z|=1}\frac{zdz}{(z^2-6z+1)^2}=16\pi\cdot\frac{1}{2\pi i}\int_{|z|=1}\frac{zdz}{(z-z_1)^2(z-z_2)^2}\\ &= 16\pi\cdot\frac{d}{dz}\left[\frac{z}{(z-z_1)^2}\right]\bigg|_{z=z_2}=16\pi\frac{z_1+z_2}{(z_1-z_2)^3}=\frac{3\pi}{4\sqrt{2}}.\end{aligned}$$

□

6. (1)

Proof. This is a special case of (3), with $n=2$ and $m=1$. See the solution there. □

(2)

Proof. Let $f(z)=\frac{1}{(1+z^2)^{n+1}}$. Then for $C_R=\{z: 0\le \arg z\le\pi, |z|=R\}$ $(R>0)$. When $R>1$, we have by Residue Theorem and Problem 2 (2) of this chapter

$$\int_{-R}^{R}f(z)dz+\int_{C_R}f(z)dz=2\pi i\text{Res}(f,i)=2\pi i\cdot\frac{-i}{2^{2n+1}}\frac{(2n)!}{(n!)^2}=\frac{\pi}{2^{2n}}\frac{(2n)!}{(n!)^2}.$$

Furthermore, we note

$$\left|\int_{C_R}f(z)dz\right|=\left|\int_0^{\pi}\frac{Re^{i\theta}id\theta}{(1+R^2e^{2i\theta})^{n+1}}\right|\le\int_0^{\pi}\frac{Rd\theta}{(R^2-1)^{n+1}}=\frac{\pi R}{(R^2-1)^{n+1}}\to 0$$

as $R\to\infty$. So $\int_{-\infty}^{\infty}\frac{1}{(1+x^2)^{n+1}}dx=\frac{\pi}{2^{2n}}\frac{(2n)!}{(n!)^2}$. □

(3)

Proof. We note

$$\int_{-\infty}^{\infty}\frac{x^{2m}}{1+x^{2n}}dx=2\int_0^{\infty}\frac{x^{2m}}{1+x^{2n}}dx=\frac{1}{n}\int_0^{\infty}\frac{y^{\frac{2m+1}{2n}}}{(1+y)y}dy$$

by the substitution rule $x^{2n}=y$. Define $p=\frac{2n}{2m+1}$, then $p>1$. In our solution of exercise 7.8 in the text, we already showed

$$\int_0^{\infty}\frac{y^{\frac{1}{p}}}{(1+y)y}dy=\pi\csc\left(\frac{\pi}{p}\right).$$

So $\int_{-\infty}^{\infty}\frac{x^{2m}}{1+x^{2m}}dx=\frac{\pi}{n}\csc\left(\frac{2m+1}{2n}\pi\right)$. □

(4)

Proof. We note $\cosh(\frac{\pi}{2}z)=0$ if and only if $z=(2n+1)i$ $(n\in\mathbb{Z})$. Define $f(z)=\frac{1}{(z^2+1)\cosh(\frac{\pi}{2}z)}$. Then for $n\neq 0,-1$, z_n is a pole of order 1 for $f(z)$ and we have (assume $\rho>0$ is sufficiently small)

$$\text{Res}(f,z_n)=\frac{1}{2\pi i}\int_{|z-z_n|=\rho}\frac{1}{z-z_n}\cdot\frac{\frac{\pi}{2}z-\frac{\pi}{2}z_n}{\cosh(\frac{\pi}{2}z)-\cosh(\frac{\pi}{2}z_n)}\frac{dz}{\frac{\pi}{2}(z^2+1)}=\frac{1}{\sinh(\frac{\pi}{2}z_n)}\cdot\frac{1}{\frac{\pi}{2}(z_n^2+1)}=\frac{i}{2\pi}\frac{(-1)^{n+1}}{n(n+1)}.$$

To find the residue of $f(z)$ at i, define $h(z)=\frac{\pi}{2}\frac{z-i}{\cosh(\frac{\pi}{2}z)-\cosh(\frac{\pi}{2}i)}$. Then $h(i)=\frac{1}{\sinh(\frac{\pi}{2}i)}=-i$. Applying l'Hospitale's rule, we have

$$\begin{aligned}h'(i) &= \lim_{z\to i}\frac{\frac{\pi}{2}\cosh(\frac{\pi}{2}z)-\frac{\pi}{2}(z-i)\frac{\pi}{2}\sinh(\frac{\pi}{2}z)}{[\cosh(\frac{\pi}{2}z)]^2}=\lim_{z\to i}\frac{\frac{\pi^2}{4}\sinh(\frac{\pi}{2}z)-\frac{\pi^2}{4}[\sinh(\frac{\pi}{2}z)+(z-i)\frac{\pi}{2}\cosh(\frac{\pi}{2}z)]}{2\cosh(\frac{\pi}{2}z)\frac{\pi}{2}\sinh(\frac{\pi}{2}z)}\\ &= \lim_{z\to i}\frac{-\frac{\pi}{2}(z-i)}{\pi\sinh(\frac{\pi}{2}z)}\cdot\frac{\pi^2}{4}=0.\end{aligned}$$

Therefore

$$\operatorname{Res}(f,i)=\frac{d}{dz}\left[\frac{2h(z)}{\pi(z+i)}\right]\bigg|_{z=i}=\frac{2[h'(z)(z+i)-h(z)]}{\pi(z+i)^2}\bigg|_{z=i}=\frac{1}{2\pi i}.$$

So

$$\sum_{n=0}^{\infty}\operatorname{Res}(f,z_n)=\frac{1}{2\pi i}+\sum_{n=1}^{\infty}\frac{i}{2\pi}\frac{(-1)^{n+1}}{n(n+1)}=\frac{1}{2\pi i}+\frac{1}{2\pi i}(-1+2\ln 2)=\frac{2\ln 2}{2\pi i}.$$

Now we consider the path $C_N=[-N,N,N+4Ni,-N+4Ni,-N]$. Then

$$\oint_{C_N} f(z)dz=I+II+III+II,$$

where

$$I=\int_{-N}^{N}f(x)dx,\ II=\int_0^{4N}f(N+iy)idy,\ III=\int_N^{-N}f(x+4Ni)dx,\ IV=\int_{4N}^{0}f(-N+iy)idy.$$

We note

$$|II|\le\int_0^{4N}\frac{dy}{(|N+iy|^2-1)\cdot\frac{e^{\pi N/2}-e^{-\pi N/2}}{2}}\le\frac{2}{e^{\frac{\pi N}{2}}-e^{-\frac{\pi N}{2}}}\cdot\frac{4N}{N^2-1}\to 0$$

as $N\to\infty$. Similarly, we can show $|IV|\to 0$ as $N\to\infty$. Meanwhile, we have

$$|III|\le\int_{-N}^{N}\frac{dx}{(|x+4Ni|^2-1)\frac{e^{\frac{\pi}{2}}+e^{-\frac{\pi}{2}}}{2}}\le\frac{2N}{16N^2-1}\to 0$$

as $N\to\infty$. Therefore, $\int_{-\infty}^{\infty}f(x)dx=\lim_{N\to\infty}\oint_{C_N}f(z)dz$ and by Residue Theorem, we have

$$\int_{-\infty}^{\infty}f(x)dx=\lim_{N\to\infty}\oint_{C_N}f(z)dz=\lim_{N\to\infty}2\pi i\sum_{n=0}^{2N-1}\operatorname{Res}(f,z_n)=2\pi i\sum_{n=0}^{\infty}\operatorname{Res}(f,z_n)=2\pi i\cdot\frac{2\ln 2}{2\pi i}=2\ln 2.$$

Remark 9. *In the proof, we used the following facts from calculus (see, for example, Shen [9], page 221):*

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}=\ln 2$$

and

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^n}{n(n+1)}&=\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{n}-\frac{1}{n+1}\right)\\&=\left(\frac{1}{2}-1\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{4}-\frac{1}{3}\right)+\cdots\\&=1-2\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots\right)\\&=1-2\ln 2.\end{aligned}$$

The calculus proof of $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}=\ln 2$ needs a little bit trick. However, if we use theory of analytic functions, then the proof becomes straightforward. Indeed, we note in the unit disc,

$$\ln(1+z)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}z^n}{n}.$$

The series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}z^n}{n}$ clearly converges at $z=1$. So by Abel's Second Theorem (see, for example, Fang [3], page 121), we must have

$$\ln 2=\lim_{z\in\mathbb{R},z\to 1}\ln(1+z)=\lim_{z\in\mathbb{R},z\to 1}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}z^n}{n}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}.$$

□

7. (1)

Proof. Let $f(z) = \frac{e^{iz}}{1+z^4}$ and $C_R = \{z : |z| = R, 0 \le \arg z \le \pi\}$. The equation $1 + z^4 = 0$ has four roots: $z_1 = e^{\frac{\pi}{4}i}$, $z_2 = e^{\frac{3\pi}{4}i}$, $z_3 = e^{-\frac{\pi}{4}i}$, and $z_4 = e^{-\frac{3\pi}{4}i}$, where z_1 and z_2 fall in the upper half plane. For R large enough, we have by Residue Theorem

$$\int_{-R}^{R} f(z)dz + \int_{C_R} f(z)dz = 2\pi i(\text{Res}(f, z_1) + \text{Res}(f, z_2)).$$

It's easy to see $\text{Res}(f, z_1) = \lim_{z\to z_1} \frac{z-z_1}{1+z^4} e^{iz} = \frac{e^{iz_1}}{4z_1^3} = -\frac{1}{4}e^{iz_1}z_1 = -\frac{1+i}{4\sqrt{2}}e^{\frac{-1+i}{\sqrt{2}}}$ and $\text{Res}(f, z_2) = -\frac{-1+i}{4\sqrt{2}}e^{\frac{-1-i}{\sqrt{2}}}$. So

$$2\pi i(\text{Res}(f, z_1) + \text{Res}(f, z_2)) = 2\pi i \cdot \frac{-1}{4\sqrt{2}} e^{-\frac{1}{\sqrt{2}}} \cdot 2i \left(\cos\frac{1}{\sqrt{2}} + \sin\frac{1}{\sqrt{2}}\right) = \frac{\pi}{\sqrt{2}} e^{-\frac{1}{\sqrt{2}}} \left(\cos\frac{1}{\sqrt{2}} + \sin\frac{1}{\sqrt{2}}\right).$$

Meanwhile, for $z \in C_R$, $\left|\frac{1}{1+z^4}\right| \le \frac{1}{R^4-1} \to 0$ as $R \to \infty$. By by Jordan's lemma (Lemma 7.1),

$$\lim_{R\to\infty} \int_{C_R} f(z)dz = 0.$$

Combined, we conclude

$$\int_0^\infty \frac{\cos x}{1+x^4}dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos x}{1+x^4}dx = \frac{1}{2}\cdot 2\pi i(\text{Res}(f, z_1) + \text{Res}(f, z_2)) = \frac{\pi}{2\sqrt{2}} e^{-\frac{1}{\sqrt{2}}} \left(\cos\frac{1}{\sqrt{2}} + \sin\frac{1}{\sqrt{2}}\right).$$

□

(2)

Proof. Let $f(z) = \frac{e^{iz}}{(1+z^2)^3}$. Then similar to our solution for part (1), we have

$$\int_0^\infty \frac{\cos x}{(1+x^2)^3}dx \quad = \quad \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos x}{(1+x^2)^3}dx = \pi i \text{Res}(f, i) = \pi i \cdot \frac{1}{2!}\frac{d^2}{dz^2}[e^{iz}(z+i)^{-3}]\,|_{z=i} = \frac{7\pi}{16e}.$$

□

(3)

Proof. Let $f(z) = \frac{ze^{iz}}{z^2-2z+2}$. Define $C_R = \{z : |z| = R, 0 \le \arg z \le \pi\}$. Then for $z \in C_R$, $\left|\frac{z}{z^2-2z+2}\right| \le \frac{R}{R^2-2R-2} \to 0$ as $R \to 0$. So by Jordan's lemma (Lemma 7.1) and Residue Theorem

$$\int_{-\infty}^{\infty} f(z)dz = 2\pi i \text{Res}(f, 1+i) = 2\pi i \cdot \left.\frac{ze^{iz}}{[z-(1-i)]}\right|_{z=1+i} = \frac{\pi}{e}[(\cos 1 - \sin 1) + i(\cos 1 + \sin 1)].$$

Compare the real and imaginary parts of both sides of the equality, we have $\int_{-\infty}^{\infty} \frac{x\sin x}{x^2-2x+2}dx = \frac{\pi}{e}(\cos 1 + \sin 1)$. □

8. (1)

Proof.

$$\begin{aligned}&\text{v.p.}\int_{-\infty}^{\infty}\frac{dx}{x(x-1)(x-2)}\\ =&\lim_{\delta\to 0}\left[\int_{-\infty}^{-\delta}\frac{dx}{x(x-1)(x_2)}+\int_{\delta}^{1-\delta}\frac{dx}{x(x-1)(x-2)}+\int_{1+\delta}^{2-\delta}\frac{dx}{x(x-1)(x-2)}+\int_{2+\delta}^{\infty}\frac{dx}{x(x-1)(x-2)}\right].\end{aligned}$$

We note $\frac{1}{x(x-1)(x-2)}=\frac{1}{2(x-2)}-\frac{1}{x-1}+\frac{1}{2x}$. So

$$\begin{aligned}\int_{-\infty}^{-\delta}\frac{dx}{x(x-1)(x-2)} &= \lim_{N\to\infty}\int_{-N}^{-\delta}\left[\frac{1}{2(x-2)}-\frac{1}{x-1}+\frac{1}{2x}\right]dx\\ &= \lim_{N\to\infty}\frac{1}{2}\ln\left[\frac{(N+1)^2}{N(N+2)}\right]+\frac{1}{2}\ln\frac{\delta(\delta+2)}{(\delta+1)^2}\\ &= \frac{1}{2}\ln\frac{\delta(\delta+2)}{(\delta+1)^2},\end{aligned}$$

$$\int_{\delta}^{1-\delta}\frac{dx}{x(x-1)(x-2)}=\int_{\delta}^{1-\delta}\left[\frac{1}{2(x-2)}-\frac{1}{x-1}+\frac{1}{2x}\right]dx=\frac{1}{2}\ln\left[\frac{(1-\delta)^3(1+\delta)}{(2-\delta)\delta^3}\right],$$

$$\int_{1+\delta}^{2-\delta}\frac{dx}{x(x-1)(x-2)}=\int_{1+\delta}^{2-\delta}\left[\frac{1}{2(x-2)}-\frac{1}{x-1}+\frac{1}{2x}\right]dx=\frac{1}{2}\ln\left[\frac{\delta}{1-\delta}\frac{\delta^2}{(1-\delta)^2}\frac{2-\delta}{1+\delta}\right],$$

and

$$\begin{aligned}\int_{2+\delta}^{\infty}\frac{dx}{x(x-1)(x-2)} &= \lim_{N\to\infty}\int_{2+\delta}^{N}\left[\frac{1}{2(x-2)}-\frac{1}{x-1}+\frac{1}{2x}\right]dx\\ &= \lim N\to\infty\frac{1}{2}\left[\frac{N-2}{\delta}\frac{(1+\delta)^2}{(N-1)^2}\frac{N}{2+\delta}\right]\\ &= \frac{1}{2}\ln\frac{(1+\delta)^2}{\delta(2+\delta)}.\end{aligned}$$

Therefore

$$\text{v.p.}\int_{-\infty}^{\infty}\frac{dx}{x(x-1)(x-2)}=\lim_{\delta\to 0}\frac{1}{2}\ln\left[\frac{\delta(\delta+2)}{(\delta+1)^2}\frac{(1-\delta)^3(1+\delta)}{(2-\delta)\delta^3}\frac{\delta^3(2-\delta)}{(1-\delta)^3(1+\delta)}\frac{(1+\delta)^2}{\delta(2+\delta)}\right]=0.$$

□

(2)

Proof. Note $\sin(x+a)\sin(x-a)=-\frac{1}{2}[\cos(2x)-\cos(2a)]$. So

$$\int_0^{\infty}\frac{\sin(x+a)\sin(x-a)}{x^2-a^2}dx=-\frac{1}{2}\int_0^{\infty}\frac{\cos(2x)-\cos(2a)}{x^2-a^2}dx=-\frac{1}{4}\int_{-\infty}^{\infty}\frac{\cos(2x)-\cos(2a)}{x^2-a^2}dx.$$

Define $f(z)=\frac{e^{2zi}-e^{2ai}}{z^2-a^2}$. Let $C_R=\{z:|z|=R,0\le\arg z\le R\}$ $(R>0)$, $c_r(a)=\{z:|z-a|=r,0\le\arg z\le\pi\}$, $c_r(-a)=\{z:|z+a|=r,0\le\arg z\le R\}$. Then by Residue Theorem,

$$\int_{(-R,-a-r)\cup c_r(-a)\cup(-a+r,a-r)\cup c_r(a)\cup(a+r,R)\cup C_R}f(z)dz=0.$$

Since a is a pole of order 1 for $f(z)$, $f(z)$ can be written as $\frac{g(z)}{z-a}$ near a where $g(z)$ is analytic near a and $g(a)\neq 0$. So

$$\int_{c_r(a)}f(z)dz=\int_{c_r(a)}\frac{g(z)}{z-a}dz=\int_{\pi}^{0}\frac{g(a+re^{i\alpha})}{re^{i\alpha}}re^{i\alpha}\cdot id\alpha=-i\int_0^{\pi}g(a+re^{i\alpha})d\alpha\to -i\pi g(a)=-i\pi\text{Res}(f,a)$$

as $r \to 0$. Similarly, $\int_{c_r(-a)} f(z)dz = -i\pi\text{Res}(f,-a)$. It's easy to see $\lim_{R\to\infty}\int_{C_R} f(z)dz = 0$. So by letting $r \to 0$ and $R \to \infty$, we have

$$\int_{-\infty}^{\infty} f(z)dz = i\pi[\text{Res}(f,a)+\text{Res}(f,-a)] = i\pi \left.\frac{e^{2zi}-e^{2ai}}{z+a}\right|_{z=a} + i\pi \left.\frac{e^{2zi}-e^{2ai}}{z-a}\right|_{z=-a} = -\frac{\pi}{a}\sin(2a).$$

So $\int_0^\infty \frac{\sin(x+a)\sin(x-a)}{x^2-a^2}dx = \frac{\pi}{4a}\sin(2a)$. □

(3)

Proof. Define $f(z) = \frac{iz-e^{iz}}{z^3(1+z^2)}$. Let $R > r > 0$, $C_R = \{z : |z| = R, 0 \le \arg z \le \pi\}$, and $C_r = \{z : |z| = r, 0 \le \arg z \le pi\}$. Then by Residue Theorem

$$\int_{-R}^{-r} f(z)dz + \int_{C_r} f(z)dz + \int_r^R f(z)dz + \int_{C_R} f(z)dz = 2\pi i[\text{Res}(f,i)+\text{Res}(f,-i)].$$

Note $\text{Res}(f,i) = \left.\frac{iz-e^{iz}}{z^3(z+i)}\right|_{z=i} = -\frac{1+e^{-1}}{2}$ and $\text{Res}(f,-i) = \left.\frac{iz-e^{iz}}{z^3(z-i)}\right|_{z=-i} = \frac{1-e}{2}$. So $\text{Res}(f,i)+\text{Res}(f,-i) = -\frac{e+e^{-1}}{2} = -\cosh 1$. Also, we note

$$\int_{C_r} f(z)dz = -i\int_0^\pi \frac{ire^{i\alpha}-e^{ire^{i\alpha}}}{r^2e^{2\alpha i}}d\alpha.$$

By repeatedly using l'Hospitale's rule, we have

$$\lim_{r\to 0}\frac{ire^{i\alpha}-e^{ire^{i\alpha}}}{r^2e^{2\alpha i}(1+r^2e^{2\alpha i})} = \lim_{r\to 0}\frac{ie^{i\alpha}-ie^{i\alpha}e^{ire^{i\alpha}}}{2re^{2\alpha i}} = \frac{1}{2}.$$

So $\lim_{r\to 0}\int_{C_r} f(z)dz = -\frac{\pi}{2}i$. It's easy to see $\lim_{R\to\infty}\int_{C_R} f(z)dz = 0$ by Jordan's lemma. Therefore, by letting $R \to \infty$ and $r \to 0$, we have

$$\int_{-\infty}^{\infty} f(z)dz = 2\pi i \cdot (-\cosh 1) + \frac{\pi}{2}i.$$

By comparing the real and imaginary parts of both sides of the equality, we obtain

$$\int_0^\infty \frac{x-\sin x}{x^3(1+x^2)}dx = \frac{1}{2}\int_{-\infty}^\infty \frac{x-\sin x}{x^3(1+x^2)}dx = \frac{\pi}{2}\left(\frac{1}{2} - e - \frac{1}{e}\right).$$

Remark 10. *The above result is different from the textbook's solution. I think I made a calculational mistake somewhere. Check.*

□

(4)

Proof. We shall use the following result: if $\alpha \ne 0$ and $(\beta/\alpha) \ne \pm 1, \pm 2, \cdots$, then

$$\frac{\pi}{\alpha}\cot\frac{\pi\beta}{\alpha} = \sum_{n=0}^{\infty}\left\{\frac{1}{n\alpha+\beta} - \frac{1}{n\alpha+(\alpha-\beta)}\right\}.$$

For a proof, see Conway [1], Chapter V, Exercise 2.8 (page 122), or my solution manual for Gong [5], Chapter 3, Exercise 11 (iii) (page 119).

We have

$$\begin{aligned}
\int_{-\infty}^{\infty}\frac{e^{px}-e^{qx}}{1-e^{x}}dx &= \int_0^{\infty}\frac{e^{-(1-p)x}-e^{-(1-q)x}}{e^{-x}-1}dx+\int_0^{\infty}\frac{e^{-py}-e^{-qy}}{1-e^{-y}}dy\\
&= \int_0^{\infty}\frac{e^{-px}-e^{-(1-p)x}}{1-e^{-x}}dx-\int_0^{\infty}\frac{e^{-qx}-e^{-(1-q)x}}{1-e^{-x}}dx\\
&= \int_0^{\infty}[e^{-px}-e^{-(1-p)x}]\sum_{n=0}^{\infty}e^{-nx}dx-\int_0^{\infty}[e^{-qx}-e^{-(1-q)x}]\sum_{n=0}^{\infty}e^{-nx}dx\\
&= \sum_{n=0}^{\infty}\int_0^{\infty}[e^{-(n+p)x}-e^{-(n+1-p)x}]dx-\sum_{n=0}^{\infty}\int_0^{\infty}[e^{-(n+q)x}-e^{-(n+1-q)x}]dx\\
&= \sum_{n=0}^{\infty}\left(\frac{1}{n+p}-\frac{1}{n+1-p}\right)-\sum_{n=0}^{\infty}\left(\frac{1}{n+q}-\frac{1}{n+1-q}\right)\\
&= \pi\cot(p\pi)-\pi\cot(q\pi).
\end{aligned}$$

□

9. A class of integration problems can be solved by the following general result (Whittaker and Watson [11], §6.24, *Evaluation of integrals of the form* $\int_0^\infty x^{\alpha-1}Q(x)dx$).

Theorem 2. *Let $Q(x)$ be a rational function of x such that it has no poles on the positive part of the real axis and $x^aQ(x)\to 0$ both when $x\to 0$ and when $x\to\infty$. If $\sum r$ denote the sum of the residues of $(-z)^{a-1}Q(z)$ at all its poles, then*

$$\int_0^{\infty}x^{a-1}Q(x)dx=\pi\csc(\alpha\pi)\sum r.$$

Corollary 1. *If $Q(x)$ has a number of simple poles on the positive part of the real axis, it may be shown by indenting the contour that*

$$v.p.\int_0^{\infty}x^{a-1}Q(x)dx=\pi\csc(a\pi)\sum r-\pi\cot(a\pi)\sum r',$$

where $\sum r'$ is the sum of the residues of $z^{a-1}Q(z)$ at these poles.

(1)

Proof. By the above theorem, we have

$$\int_0^{\infty}\frac{x^{s-1}}{1-x}dx=-\pi\cot(s\pi)\mathrm{Res}\left(\frac{x^{s-1}}{1-x},1\right)=\pi\cot(s\pi).$$

□

(2)

Proof. If $s=1$, then

$$\int_0^{\infty}\frac{xdx}{(1+x^2)^2}=\frac{1}{2}\int_0^{\infty}\frac{dy}{(1+y)^2}=-\frac{1}{2(1+y)}\bigg|_0^{\infty}=\frac{1}{2}.$$

To calculate the case where $s\neq 1$, we choose r and R such that $0<r<R$. Let $\gamma_1=\{z:r\le|z|\le R,\arg z=0\}$, $\gamma_2=\{z:r\le|z|\le R,\arg z=\pi\}$, $\gamma_R=\{z:|z|=R,0<\arg z<\pi\}$, and $\gamma_r=\{z:|z|=r,0<\arg z<\pi\}$. Define $f(z)=\frac{z^s}{(1+z^2)^2}$. Suppose r is sufficiently small and R is sufficiently large so that all the poles of $f(z)$ are contained in the contour formed by γ_1, γ_2, γ_r, and γ_R. Then

$$\int_{\gamma_1+\gamma_R+\gamma_2-\gamma_r}f(z)dz=2\pi i\mathrm{Res}(f,i)=2\pi i\cdot\frac{d}{dz}\frac{z^s}{(z+i)^2}\bigg|_{z=i}=2\pi i\cdot\frac{s-1}{-4}e^{\frac{\pi}{2}(s-1)i}=-\frac{\pi}{2}(s-1)e^{\frac{\pi}{2}si}.$$

We have (note $-1 < s < 3$)

$$\left|\int_{\gamma_R} f(z)dz\right| = \left|\int_0^\pi \frac{(Re^{i\theta})^s}{(1+R^2e^{2i\theta})^2}Re^{i\theta}\cdot id\theta\right| \le \frac{\pi R^{s+1}}{(R^2-1)^2} \to 0$$

as $R \to \infty$,

$$\left|\int_{\gamma_r} f(z)dz\right| = \left|\int_0^\pi \frac{(re^{i\theta})^s}{(1+r^2e^{2i\theta})^2}re^{i\theta}\cdot id\theta\right| \le \frac{\pi r^{s+1}}{(1-r^2)^2} \to 0$$

as $r \to 0$, and

$$\int_{\gamma_2} f(z)dz = \int_0^\infty \frac{(xe^{\pi i})^s}{(1+x^2)^2}dx = e^{s\pi i}\int_0^\infty \frac{x^s}{(1+x^2)^2}dx.$$

Since $s \ne 1$, $e^{s\pi i} \ne -1$. Therefore

$$\int_0^\infty \frac{x^s}{(1+x^2)^2}dx = \frac{2\pi i \text{Res}(f,i)}{1+e^{s\pi i}} = \frac{\pi}{4}\frac{1-s}{\cos\left(\frac{\pi}{2}s\right)}.$$

Combining all the cases and regarding the cases where $s \in \mathbb{Z}$ as limit case of the formula $\frac{\pi}{4}\frac{1-s}{\cos(\frac{\pi}{2}s)}$, we conclude the integral is evaluated to $\frac{\pi}{4}\frac{1-s}{\cos(\frac{\pi}{2}s)}$.

Remark 11. *We could have used the general theorem, but we still go to the specific solution so that some insight can be shed on how the general theorem is proved.*

□

(3)

Proof. We choose r and R such that $0 < r < R$. Let $\gamma_1 = \{z : r \le |z| \le R, \arg z = 0\}$, $\gamma_2 = \{z : r \le |z| \le R, \arg z = 2\pi\}$, $\gamma_R = \{z : |z| = R, 0 < \arg z < 2\pi\}$, and $\gamma_r = \{z : |z| = r, 0 < \arg z < 2\pi\}$. Define $f(z) = \frac{z^{\alpha-1}\ln z}{1+z}$. Suppose r is sufficiently small and R is sufficiently large so that all the poles of $f(z)$ are contained in the contour formed by γ_1, γ_2, γ_r, and γ_R. Then by Residue Theorem

$$\int_{\gamma_1+\gamma_R-\gamma_2-\gamma_r} f(z)dz = 2\pi i \text{Res}(f,-1) = 2\pi i(-1)^{\alpha-1}\ln(-1) = -2\pi^2 e^{(\alpha-1)\pi i}.$$

We note

$$\left|\int_{\gamma_R} f(z)dz\right| \le \left|\int_0^{2\pi} \frac{(Re^{i\theta})^{\alpha-1}\ln(Re^{i\theta})}{1+Re^{i\theta}}Re^{i\theta}\cdot id\theta\right| \le \frac{2\pi R^\alpha(\ln R + 2\pi)}{R-1} \to 0$$

as $R \to \infty$,

$$\left|\int_{\gamma_r} f(z)dz\right| \le \left|\int_0^{2\pi} \frac{(re^{i\theta})^{\alpha-1}\ln(re^{i\theta})}{1+re^{i\theta}}re^{i\theta}\cdot id\theta\right| \le \frac{2\pi r^\alpha(\ln r + 2\pi)}{1-r} \to 0$$

as $r \to 0$, and

$$\begin{aligned}\int_{\gamma_2} f(z)dz &= \int_r^R \frac{(xe^{2\pi i})^{\alpha-1}\ln(xe^{2\pi i})}{1+x}dx \\ &= \int_r^R \frac{x^{\alpha-1}e^{2(\alpha-1)\pi i}(\ln x + 2\pi i)}{1+x}dx \\ &= e^{2(\alpha-1)\pi i}\int_r^R \frac{x^{\alpha-1}\ln x}{1+x}dx + e^{2(\alpha-1)\pi i}2\pi i\int_r^R \frac{x^{\alpha-1}}{1+x}dx.\end{aligned}$$

It's not hard to show $\int_0^\infty \frac{x^{\alpha-1}}{1+x}dx = \pi\csc(\alpha\pi)$. So by letting $r\to 0$ and $R\to\infty$, we have

$$\begin{aligned}\int_0^\infty \frac{x^{\alpha-1}\ln x}{1+x}dx &= \frac{-2\pi^2 e^{(\alpha-1)\pi i}+e^{2(\alpha-1)\pi i}2\pi i\cdot\pi\csc(\alpha\pi)}{1-e^{2(\alpha-1)\pi i}}\\ &= \frac{2\pi^2 e^{\alpha\pi i}+2\pi^2\csc(\alpha\pi)e^{2\alpha\pi i}i}{1-e^{2\alpha\pi i}}\\ &= 2\pi^2\frac{1+\csc(\alpha\pi)i[\cos(\alpha\pi)+i\sin(\alpha\pi)]}{e^{-\alpha\pi i}-e^{\alpha\pi i}}\\ &= -\pi^2\frac{\cos(\alpha\pi)}{\sin^2(\alpha\pi)}.\end{aligned}$$

□

8 Γ Function

1. (1)

Proof. $(2n)!! = (2n)\cdot(2n-2)\cdots 2 = 2^n\cdot n! = 2^n\Gamma(n+1)$. □

(2)

Proof. $(2n-1)!! = \frac{2n\cdot(2n-1)\cdot(2n-2)\cdots 3\cdot 2\cdot 1}{2n\cdot(2n-2)\cdots 2} = \frac{(2n)!}{2^n\cdot n!} = \frac{\Gamma(2n+1)}{2^n\Gamma(n+1)}$. □

(3)

Proof. $\Gamma(n+v+1) = (n+v)\Gamma(n+v) = \cdots = (n+v)(n-1+v)\cdots(1+v)\Gamma(1+v)$. So $(1+v)(2+v)\cdots(n+v) = \frac{\Gamma(n+v+1)}{\Gamma(v+1)}$. □

(4)

Proof.

$$\begin{aligned}\prod_{l=0}^{n}[l(l+1)-v(v+1)] &= \prod_{l=0}^{n}[(l-v)(l+v+1)]\\ &= \prod_{l=0}^{n}(l-v)\cdot\prod_{l=0}^{n}(l+v+1)\\ &= \frac{\Gamma(n-v+1)}{\Gamma(-v)}\cdot\frac{\Gamma(n+v+2)}{\Gamma(v+1)}\\ &= \frac{\Gamma(n+v+2)\Gamma(n-v+1)}{\frac{\pi}{\sin\pi(v+1)}}\\ &= -\frac{\sin\pi v}{\pi}\Gamma(n+v+2)\Gamma(n-v+1).\end{aligned}$$

□

2. (1)

Proof. We first assume $\alpha\in(0,1)$. Let $C_R = \{z : |z| = R, 0\le \arg z\le \frac{\pi}{2}\}$, $C_r = \{z : |z| = r, 0\le\arg z\le\frac{\pi}{2}\}$, and assume $R > r$. Then by Residue Theorem

$$\int_r^R \frac{e^{iz}}{z^\alpha}dz + \int_{C_R}\frac{e^{iz}}{z^\alpha}dz + \int_{iR}^{ir}\frac{e^{iz}}{z^\alpha}dz + \int_{C_r}\frac{e^{iz}}{z^\alpha}dz = 0.$$

Then it's easy to see

$$\left|\int_{C_r} \frac{e^{iz}}{z^\alpha} dz\right| \leq \left|\int_{\frac{\pi}{2}}^{0} \frac{e^{ire^{i\theta}}}{(re^{i\theta})^\alpha} re^{i\theta} \cdot id\theta\right| \leq \int_0^{\frac{\pi}{2}} r^{1-\alpha} e^{-r\sin\theta} d\theta \leq \frac{\pi}{2} r^{1-\alpha} \to 0$$

as $r \to 0$, and

$$\left|\int_{C_R} \frac{e^{iz}}{z^\alpha} dz\right| \leq \left|\int_0^{\frac{\pi}{2}} \frac{e^{iRe^{i\theta}}}{(Re^{i\theta})^\alpha} Re^{i\theta} \cdot id\theta\right| \leq \int_0^{\frac{\pi}{2}} R^{1-\alpha} e^{-R\sin\theta} d\theta \leq \int_0^{\frac{\pi}{2}} R^{1-\alpha} R^{-R\frac{2\theta}{\pi}} d\theta = \frac{\pi}{2R^\alpha}(1 - e^{-R}) \to 0$$

as $R \to \infty$. So by letting $r \to 0$ and $R \to \infty$, we have

$$\int_0^\infty \frac{e^{iz}}{z^\alpha} dz = \int_0^\infty \frac{e^{i(ix)}}{(ix)^\alpha} d(ix) = i \cdot (-i)^\alpha \int_0^\infty \frac{e^{-x}}{x^\alpha} dx = ie^{-\frac{\pi}{2}\alpha i}\Gamma(1-\alpha).$$

This implies

$$\int_0^\infty x^{-\alpha} \cos x dx + i \int_0^\infty x^{-\alpha} \sin x dx = \left(\sin \frac{\alpha}{2}\pi + i \cos \frac{\alpha}{2}\pi\right) \Gamma(1-\alpha).$$

Compare and equal the real and imaginary parts of the two sides, we get $\int_0^\infty x^{-\alpha} \sin x dx = \Gamma(1-\alpha) \cos \frac{\alpha}{2}\pi$ and $\int_0^\infty x^{-\alpha} \cos x dx = \Gamma(1-\alpha) \sin \frac{\alpha}{2}\pi$. For $\alpha \in (1,2)$, we note

$$\int_0^\infty x^{-(\alpha-1)} \cos x dx = x^{-(\alpha-1)} \sin x \,|_0^\infty + (\alpha-1) \int_0^\infty x^{-\alpha} \sin x dx = (\alpha-1) \int_0^\infty x^{-\alpha} \sin x dx.$$

So for $\alpha \in (1,2)$, $\int_0^\infty x^{-\alpha} \sin x dx = \frac{1}{\alpha-1}\Gamma(1-(\alpha-1)) \sin \frac{\alpha-1}{2}\pi = \Gamma(1-\alpha)\cos\frac{\alpha}{2}\pi$. That is, the formula for $\int_0^\infty x^{-\alpha} \sin x dx$ is the same when $\alpha \in (0,1)$ and $\alpha \in (1,2)$. When $\alpha = 1$, Example 7.9 in the textbook shows $\int_0^\infty \frac{\sin x}{x} dx = \pi$, which cannot be obtained by plugging $\alpha = 1$ into $\Gamma(1-\alpha)\cos\frac{\alpha}{2}\pi$. □

(2)

Proof. Let $C_R = \{z : |z| = R, 0 \leq \arg z \leq \theta\}$, $C_r = \{z : |z| = r, 0 \leq \arg z \leq \theta\}$ and assume $R > r$. Then by Residue Theorem

$$\int_r^R z^{\alpha-1}e^{-z}dz + \int_{C_R} z^{\alpha-1}e^{-z}dz + \int_R^r (xe^{i\theta})^{\alpha-1}e^{-xe^{i\theta}}d(xe^{i\theta}) + \int_{C_r} z^{\alpha-1}e^{-z}dz = 0.$$

Note cos function is positive on $(-\frac{\pi}{2}, \frac{\pi}{2})$, we have

$$\left|\int_{C_R} z^{\alpha-1}e^{-z}dz\right| = \left|\int_0^\theta (Re^{i\xi})^{\alpha-1}e^{-Re^{i\xi}}d(Re^{i\xi})\right| \leq \int_0^\theta R^{\alpha-1}e^{-R\cos\xi} \cdot Rd\xi = \int_0^\theta R^\alpha e^{-R\cos\xi}d\xi.$$

On the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, $\cos\xi = \sin(\xi + \frac{\pi}{2}) \geq \frac{2}{\pi}(\xi + \frac{\pi}{2})$. So

$$\left|\int_{C_R} z^{\alpha-1}e^{-z}dz\right| \leq \int_0^\theta R^\alpha e^{-\frac{2R}{\pi}(\xi+\frac{\pi}{2})}d\xi = \frac{\pi R^{\alpha-1}}{2}\left[e^{-R} - e^{-R(1+\frac{2}{\pi}\theta)}\right].$$

Since $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $1 + \frac{2}{\pi}\theta > 0$. So $\lim_{R\to\infty} \frac{\pi R^{\alpha-1}}{2}\left[e^{-R} - e^{-R(1+\frac{2}{\pi}\theta)}\right] = 0$. Also, we note

$$\left|\int_{C_r} z^{\alpha-1}e^{-z}dz\right| \leq \int_0^\theta r^\alpha e^{-R\cos\xi}d\xi \leq \theta r^\alpha \to 0$$

as $r \to 0$. So by letting $r \to 0$ and $R \to \infty$, we have

$$\begin{aligned}\int_0^\infty x^{\alpha-1}e^{-x}dx &= \int_0^\infty x^{\alpha-1}e^{i\theta(\alpha-1)}e^{-xe^{i\theta}}e^{i\theta}dx = \int_0^\infty x^{\alpha-1}e^{i\theta\alpha}e^{-xe^{i\theta}}dx \\ &= e^{i\theta\alpha}\int_0^\infty x^{\alpha-1}e^{-x\cos\theta}[\cos(x\sin\theta) - i\sin(x\sin\theta)]dx.\end{aligned}$$

Let $I=\int_0^\infty x^{\alpha-1}e^{-x\cos\theta}\cos(x\sin\theta)dx$ and $II=\int_0^\infty x^{\alpha-1}e^{-x\cos\theta}\sin(x\sin\theta)dx$. Then we have

$$\Gamma(\alpha)=(\cos\theta\alpha+i\sin\theta\alpha)(I-iII)=(I\cos\theta\alpha+II\sin\theta\alpha)+i(I\sin\theta\alpha-II\cos\theta\alpha).$$

Equating the real and imaginary parts of the terms on both sides of the equation, we can get two equations of I and II. Solving these two equations gives us

$$I=\Gamma(\alpha)\cos\alpha\theta,\ II=\Gamma(\alpha)\sin\alpha\theta.$$

□

3. (1)

Proof. Since $\Gamma(z+1)=z\Gamma(z)$, we have $\Gamma'(z+1)=\Gamma(z)+z\Gamma'(z)$. So

$$\Psi(z+1)=\frac{\Gamma'(z+1)}{\Gamma(z+1)}=\frac{\Gamma(z)+z\Gamma'(z)}{z\Gamma(z)}=\frac{1}{z}+\Psi(z).$$

□

(2)

Proof. $\Psi(z+n)=\frac{1}{z+n-1}+\Psi(z+n-1)=\frac{1}{z+n-1}+\frac{1}{z+n-2}+\Psi(z+n-2)=\cdots=\frac{1}{z+n-1}+\frac{1}{z+n-2}+\cdots+\frac{1}{z}+\Psi(z)$. □

(3)

Proof. By $\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin\pi z}$, we have $\ln\Gamma(z)+\ln\Gamma(1-z)=\ln\pi-\ln(\sin\pi z)$. Differentiating both sides, we have $\Psi(z)-\Psi(1-z)=-\frac{\cos\pi z}{\sin\pi z}\cdot\pi$. So $\Psi(1-z)-\Psi(z)=\pi\cot\pi z$. □

(4)

Proof. By the formula $\Gamma(2z)=2^{2z-1}\pi^{-\frac{1}{2}}\Gamma(z)\Gamma(z+\frac{1}{2}$, we have

$$\ln\Gamma(2z)=(2z-1)\ln 2-\frac{1}{2}\ln\pi+\ln\Gamma(z)+\ln\Gamma(z+\frac{1}{2}).$$

Differentiating both sides, we get $2\Psi(2z)=2\ln+\Psi(z)+\Psi(z+\frac{1}{2})$. □

4. (1)

Proof. Use the substitution $x=2y-1$, we get

$$\int_{-1}^1(1-x)^p(1+x)^q dx=\int_0^1(2-2y)^p(2y)^q\cdot 2dy=2^{p+q+1}\int_0^1(1-y)^p y^q dy=2^{p+q+1}\mathrm{B}(p+1,q+1).$$

□

(2)

Proof. Let $p=\frac{1+\alpha}{2}$ and $q=\frac{1-\alpha}{2}$. Then $p,q>0$, $p+q=1$, and

$$\int_0^{\frac{\pi}{2}}\tan^\alpha\theta d\theta=\int_0^{\frac{\pi}{2}}\sin^\alpha\theta\cos^{-\alpha}\theta d\theta=\int_0^{\frac{\pi}{2}}\sin^{2p-1}\theta\cos^{2q-1}\theta d\theta=\frac{1}{2}\mathrm{B}(p,q)=\frac{\pi}{2\sin\pi q}=\frac{\pi}{2\cos\frac{\alpha\pi}{2}}.$$

□

5. (1)

Proof. We note

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n(4n^2-1)} &= \sum_{n=1}^{\infty}\frac{1}{2}\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)=\sum_{n=1}^{\infty}\left(\frac{1}{2n-1}-\frac{1}{2n}-\frac{1}{2n}+\frac{1}{2n+1}\right)\\ &= \sum_{n=1}^{\infty}\frac{1}{2}\left(\frac{1}{n-\frac{1}{2}}-\frac{2}{n}+\frac{1}{n+\frac{1}{2}}\right).\end{aligned}$$

By formula (8.27), we conclude $\sum_{n=1}^{\infty}\frac{1}{n(4n^2-1)}=-\frac{1}{2}\left[\Psi(-\frac{1}{2})-2\Psi(0)+\Psi(\frac{1}{2})\right]$. By the formula $\Psi(2z)=\frac{1}{2}\Psi(z)+\frac{1}{2}\Psi(z+\frac{1}{2})+\ln 2$, we get $\Psi(0)=\Psi(\frac{1}{2})+2\ln 2=-\gamma$, where γ is the Euler constant. This implies

$$\sum_{n=1}^{\infty}\frac{1}{n(4n^2-1)}=-\frac{1}{2}\left[(-\gamma-2\ln 2+2)-2(-\gamma)+(-\gamma-2\ln 2)\right]=2\ln 2-1.$$

□

(2)

Proof. Using Mathematica command **Apart[1/(z^ 2 - a^ 2)^ 2**, we have

$$\begin{aligned}\sum_{n=-\infty}^{\infty}\frac{1}{(n^2+1)^2} &= 2\sum_{n=0}^{\infty}\frac{1}{(n^2+1)^2}-1=2\sum_{n=0}^{\infty}\left[-\frac{1}{4(z-i)^2}-\frac{i}{4(z-i)}-\frac{1}{4(z+i)^2}+\frac{i}{4(z+i)}\right]-1\\ &= -\frac{1}{2}\sum_{n=0}^{\infty}\left[\frac{i}{z-i}+\frac{1}{(z-i)^2}-\frac{i}{z+i}+\frac{1}{(z+i)^2}\right]-1.\end{aligned}$$

By formula (8.29c),

$$\sum_{n=-\infty}^{\infty}\frac{1}{(n^2+1)^2}=-\frac{1}{2}\cdot(-1)\cdot\left[i\Psi(-i)-\Psi'(-i)-i\Psi(i)-\Psi'(i)\right]-1.$$

By the formula $\Psi(z)-\Psi(-z)=-\frac{1}{z}-\pi\cot\pi z$, we have $\Psi(-i)-\Psi(i)=-\frac{1}{-i}-\pi\cot(-\pi i)=i(-1-\pi\coth\pi)$. And by $\Psi'(z)+\Psi'(-z)=\frac{1}{z^2}-\pi^2\csc^2(\pi z)$, we have $\Psi'(i)+\Psi'(-i)=-1-\pi^2\csc^2(i\pi)=-1-\frac{\pi^2}{\sinh^2\pi}$. Therefore

$$\sum_{n=-\infty}^{\infty}\frac{1}{(n^2+1)^2}=\frac{1}{2}\left(1+\pi\cot\pi z+1+\frac{\pi^2}{\sinh^2\pi}\right)-1=\frac{\pi}{2}\coth\pi+\frac{\pi^2}{2\sinh^2\pi}.$$

□

9 Laplace Transform

9.1 Exercise in the text

9.1.

Proof.

$$\mathcal{L}\{f(t-\tau)\}=\int_0^{\infty}e^{-pt}f(t-\tau)\eta(t-\tau)dt=e^{-p\tau}\int_0^{\infty}e^{-p(t-\tau)}f(t-\tau)\eta(t-\tau)dt=e^{-p\tau}F(p).$$

$$\mathcal{L}\{f(at)\}=\int_0^{\infty}e^{-pt}f(at)dt=\frac{1}{a}\int_0^{\infty}e^{-\frac{p}{a}(\xi)}f(\xi)d\xi=\frac{1}{a}F(\frac{p}{a}).$$

$$\mathcal{L}\{e^{p_0t}f(t)\}=\int_0^{\infty}e^{-(p-p_0)t}f(t)dt=F(p-p_0).$$

□

9.2.

Proof.

$$\mathcal{L}\left\{\int_0^\infty f(t,\tau)d\tau\right\} = \int_0^\infty e^{-pt}\int_0^\infty f(t,\tau)d\tau dt = \int_0^\infty d\tau\int_0^\infty e^{-pt}f(t,\tau)dt = \int_0^\infty F(p,\tau)d\tau.$$

Define $g(t) = \int_t^\infty \frac{f(\tau)}{\tau}d\tau$, then $g'(t) = -\frac{f(t)}{t}$. By Property 4, $\mathcal{L}\{g'(t)\} = p\mathcal{L}\{g(t)\} - g(0)$. So $-\int_0^\infty e^{-pt}\frac{f(t)}{t}dt = p\mathcal{L}\{g(t)\} - \int_0^\infty \frac{f(\tau)}{\tau}d\tau$, which implies

$$\mathcal{L}\{g(t)\} = \frac{1}{p}\int_0^\infty \frac{1-e^{-pt}}{t}f(t)dt = \frac{1}{p}\int_0^\infty\int_0^p e^{-qt}dqf(t)dt = \frac{1}{p}\int_0^p\int_0^\infty e^{-qt}f(t)dtdq = \frac{1}{p}\int_0^p F(q)dq.$$

□

9.2 Exercise at the end of chapter

1. (1)

Proof. $F_n(p) = \mathcal{L}\{t^n\} = \frac{n!}{p^{n+1}}$ for $p \in \mathbb{C}$ with Re$p > 0$. To prove this, we work by induction. When $n = 0$, this is just Example 9.1. Assume the formula is true for $k = 0, 1, \cdots, n$. Then

$$F_{n+1}(p) = \mathcal{L}\{t^{n+1}\} = \int_0^\infty e^{-pt}t^{n+1}dt = -\frac{1}{p}\left(e^{-pt}t^{n+1}|_0^\infty - \int_0^\infty e^{-pt}(n+1)t^n dt\right) = \frac{n+1}{p}F_n(p) = \frac{(n+1)!}{p^{n+2}}.$$

Here we have used Re$p > 0$ to conclude $e^{-pt}t^{n+1}|_0^\infty = 0$. By induction, we proved our claim. □

(2)

Proof. $F(p) = \frac{\Gamma(\alpha+1)}{p^{\alpha+1}}$ for $p \in \mathbb{C}$ with Re$p > 0$. Indeed,

$$\mathcal{L}\{t^\alpha\} = \int_0^\infty e^{-pt}t^\alpha dt = \frac{1}{p^{\alpha+1}}\int_0^\infty e^{-pt}(pt)^\alpha d(pt) = \frac{1}{p^{\alpha+1}}\int_L e^{-t}t^{(\alpha+1)-1}dt,$$

where L is the radial straight line that goes from 0 to ∞, with angle $\arg p$. By the extended definition of Γ function (8.3), we have $F(p) = \frac{\Gamma(\alpha+1)}{p^{\alpha+1}}$ (Re$p > 0$). □

(3)

Proof. The problem and its solution in the textbook do not match. So we calculate the Laplace transform both for $e^{\lambda t}\sin\omega t$ and $e^{-\lambda t}\sin\omega t$.

$$\mathcal{L}\{e^{-\lambda t}\sin\omega t\} = \int_0^\infty e^{-pt}e^{-\lambda t}\sin\omega t dt = \int_0^\infty e^{-(p+\lambda)t}\sin\omega t dt = \frac{\omega}{(p+\lambda)^2+\omega^2}$$

where we require Re$p > -\lambda$, and

$$\mathcal{L}\{e^{\lambda t}\sin\omega t\} = \int_0^\infty e^{-pt}e^{\lambda t}\sin\omega t dt = \int_0^\infty e^{-(p-\lambda)t}\sin\omega t dt = \frac{\omega}{(p-\lambda)^2+\omega^2}$$

where we require Re$p > \lambda$. □

(4)

Proof. By the formula $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_p^\infty F(q)dq$, we have

$$\mathcal{L}\left\{\frac{\sin\omega t}{t}\right\} = \int_p^\infty \mathcal{L}\{\sin\omega t\}dq = \int_p^\infty \frac{\omega}{q^2+\omega^2}dq = \arctan x|_{\frac{p}{\omega}}^\infty = \frac{\pi}{2} - \arctan\frac{p}{\omega} = \arctan\frac{\omega}{p},$$

where we require Re$p > 0$.

Remark 12. *The above result differs from the textbook's solution, but matches with the result of* **Mathematica**.

□

(5)

Proof. By applying the formula $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_p^\infty F(q)dq$ twice, we have

$$\begin{aligned}
\mathcal{L}\left\{\frac{1-\cos\omega t}{t^2}\right\} &= \int_p^\infty \mathcal{L}\left\{\frac{1-\cos\omega t}{t}\right\} dq = \int_p^\infty \int_q^\infty \mathcal{L}\{1-\cos\omega t\} dr dq \\
&= \int_p^\infty \int_q^\infty \left(\frac{1}{r} - \frac{r}{r^2+\omega^2}\right) dr dq \\
&= \int_p^\infty \lim_{N\to\infty} \left[\ln N - \ln q - \frac{1}{2}\ln(N^2+\omega^2) + \frac{1}{2}\ln(q^2+\omega^2)\right] dq \\
&= \lim_{N\to\infty} \int_p^N \left[\frac{1}{2}\ln(q^2+\omega^2) - \ln q\right] dq \\
&= \lim_{N\to\infty} N \ln \frac{\sqrt{N^2+\omega^2}}{N} + \int_p^\infty \frac{\omega^2}{q^2+\omega^2} dq + p\ln p - \frac{1}{2} p \ln(p^2+\omega^2) \\
&= w \arctan\frac{\omega}{p} - \frac{p}{2}\ln\frac{p^2+\omega^2}{p^2}.
\end{aligned}$$

□

(6)

Proof. We require p satisfy $\text{Re}p > 0$. Then

$$\mathcal{L}\left\{\int_t^\infty \frac{\cos\tau}{\tau} d\tau\right\} = \lim_{\delta\to 0}\int_\delta^\infty e^{-pt}\left(\int_t^\infty \frac{\cos\tau}{\tau}d\tau\right) dt.$$

By integration-by-parts formula, we have

$$\begin{aligned}
\int_\delta^\infty e^{-pt}\left(\int_t^\infty \frac{\cos\tau}{\tau}d\tau\right) dt &= -\frac{1}{p}\left[e^{-pt}\int_t^\infty \frac{\cos\tau}{\tau}d\tau\Big|_\delta^\infty + \int_\delta^\infty e^{-pt}\frac{\cos t}{t}dt\right] \\
&= \frac{1}{p}\int_\delta^\infty (e^{-p\delta} - e^{-p\tau})\frac{\cos\tau}{\tau}d\tau.
\end{aligned}$$

So by the formula $\int_0^\infty F(p)dp = \int_0^\infty \frac{f(t)}{t}dt$, we have

$$\begin{aligned}
\mathcal{L}\left\{\int_t^\infty \frac{\cos\tau}{\tau}d\tau\right\} &= \frac{1}{p}\int_0^\infty (1-e^{-pt})\frac{\cos t}{t}dt = \frac{1}{p}\int_0^\infty \mathcal{L}\{(1-e^{-pt})\cos t\} dq \\
&= \frac{1}{p}\int_0^\infty \left[\frac{q}{q^2+1} - \mathcal{L}\{e^{-pt}\cos t\}\right] dq \\
&= \frac{1}{p}\int_0^\infty \left[\frac{q}{q^2+1} - \frac{p+q}{(p+q)^2+1}\right] dq \\
&= \lim_{N\to\infty}\frac{1}{p}\int_0^N \left[\frac{1}{2}d\ln(q^2+1) - \frac{1}{2}d\ln((p+q)^2+1)\right] \\
&= \frac{1}{2p}\ln\frac{q^2+1}{(q+p)^2+1}\Big|_{q=0}^\infty \\
&= \frac{1}{2p}\ln(p^2+1).
\end{aligned}$$

Remark 13. *If we apply the result of Exercise 9.2, $\int_t^\infty \frac{f(\tau)}{\tau}d\tau \risingdotseq \frac{1}{p}\int_0^p F(q)dq$, the calculation is only one step. The function $-\int_t^\infty \frac{\cos\tau}{\tau}d\tau$ is called cosine integral function.*

□

2.

Proof.

$$\begin{aligned} F(p) &= \int_0^\infty e^{-pt}f(t)dt = \sum_{n=0}^\infty \int_{n\alpha}^{(n+1)\alpha} e^{-pt}f(t)dt = \sum_{n=0}^\infty \int_0^\alpha e^{-p(t+n\alpha)}f(t+n\alpha)dt \\ &= \sum_{n=0}^\infty \int_0^\alpha e^{-pt}f(t)dt\cdot e^{-\alpha pn} = \int_0^\alpha e^{-pt}f(t)dt\sum_{n=0}^\infty \left(e^{-\alpha p}\right)^n = \frac{1}{1-e^{-\alpha p}}\int_0^\alpha e^{-pt}f(t)dt. \end{aligned}$$

□

3. (1)

Proof. $|\sin\omega t|$ has period $\frac{\pi}{\omega}$. Using result of Problem 2, we have

$$\mathcal{L}\{|\sin\omega t|\} = \frac{1}{1-e^{-\frac{\pi}{\omega}p}}\int_0^{\frac{\pi}{\omega}} e^{-pt}\sin\omega t dt.$$

By applying integration-by-parts formula twice, we can easily verify

$$\int_0^{\frac{\pi}{\omega}} e^{-pt}\sin\omega t dt = \frac{\omega(1+e^{-\frac{\pi}{\omega}p})}{p^2+\omega^2}.$$

So

$$\mathcal{L}\{|\sin\omega t|\} = \frac{(1+e^{-\frac{\pi}{\omega}p})}{1-e^{-\frac{\pi}{\omega}p}}\frac{\omega}{p^2+\omega^2} = \frac{\omega}{p^2+\omega^2}\coth\frac{p\pi}{2\omega}.$$

□

(2)

Proof. $f(t) = t - a\left[\frac{t}{a}\right]$ has period a. So by Problem 2, we have

$$\mathcal{L}\left\{t-a\left[\frac{t}{a}\right]\right\} = \frac{1}{1-e^{-ap}}\int_0^a e^{-pt}\left(t-a\left[\frac{t}{a}\right]\right)dt = \frac{1}{1-e^{-ap}}\int_0^a e^{-pt}tdt = \frac{1}{p^2} - \frac{a}{p}\frac{e^{-ap}}{1-e^{-ap}}.$$

□

4. (1)

Proof. By the formula $\mathcal{L}\{(-t)^n f(t)\} = [\mathcal{L}\{f(t)\}]^{(n)}$, we have

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{a^3}{p(p+a)^3}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{p} - \frac{a^2}{(a+p)^3} - \frac{a}{(a+p)^2} - \frac{1}{a+p}\right\} \\ &= \eta(t) - \frac{a^2}{2}\mathcal{L}^{-1}\left\{\left(\frac{1}{p+a}\right)^{(2)}\right\} + a\mathcal{L}^{-1}\left\{\left(\frac{1}{p+a}\right)'\right\} - e^{-at}\eta(t) \\ &= \eta(t) - \frac{a^2}{2}(-t)^2e^{-at}\eta(t) + a(-t)e^{-at}\eta(t) - e^{-at}\eta(t) \\ &= \left[1 - e^{-at}\left(1+at+\frac{a^2t^2}{2}\right)\right]\eta(t). \end{aligned}$$

□

(2)

Proof. We note $\frac{\omega}{p(p^2+\omega^2)} = \frac{1}{\omega p} - \frac{p}{\omega(\omega^2+p^2)}$. So $\mathcal{L}^{-1}\left\{\frac{\omega}{p(p^2+\omega^2)}\right\} = \frac{1}{\omega}(1-\cos\omega t)\eta(t)$. □

(3)

Proof. We note $\frac{4p-1}{(p^2+p)(4p^2-1)} = \frac{1}{p} + \frac{5}{3}\frac{1}{p+1} + \frac{1}{3}\frac{1}{p-\frac{1}{2}} - \frac{3}{p+\frac{1}{2}}$. Therefore

$$\mathcal{L}^{-1}\left\{\frac{4p-1}{(p^2+p)(4p^2-1)}\right\} = \left(1 + \frac{5}{3}e^{-t} + \frac{1}{3}e^{t/2} - 3e^{-t/2}\right)\eta(t).$$

□

(4)

Proof. We note

$$\frac{p^2+\omega^2}{(p^2-\omega^2)^2} = \frac{1}{2(p-\omega)^2} + \frac{1}{2(p+\omega)^2} = -\frac{1}{2}[\mathcal{L}\{e^{\omega t}\}]' - \frac{1}{2}[\mathcal{L}\{e^{-\omega t}\}]' = (\mathcal{L}\{-\cosh\omega t\})'.$$

So by the formula $\mathcal{L}\{(-t)^n f(t)\} = [\mathcal{L}\{f(t)\}]^{(n)}$, we have

$$\mathcal{L}\left\{\frac{p^2+\omega^2}{(p^2-\omega^2)^2}\right\} = (-t)(-\cosh\omega t)\eta(t) = t\cosh(\omega t)\eta(t).$$

□

(5)

Proof. We note $\mathcal{L}\{1_{t\geq\tau}\} = \frac{e^{-p\tau}}{p}$. So by the formula $\mathcal{L}\{\int_0^t f(s)ds\} = \frac{F(p)}{p}$, we have

$$\mathcal{L}^{-1}\{\frac{e^{-p\tau}}{p^2}\} = \int_0^t 1_{\{s\geq\tau\}}ds = (t-\tau)\eta(t-\tau).$$

□

(6)

Proof. We have shown in Problem 3(2) that

$$\mathcal{L}\left\{t - \alpha\left[\frac{t}{\alpha}\right]\right\} = \frac{1}{p^2} - \frac{\alpha}{p}\frac{e^{-\alpha p}}{1-e^{-\alpha p}}.$$

By the formula $\mathcal{L}\{(-t)^n f(t)\} = [\mathcal{L}\{f(t)\}]^{(n)}$, we have $\mathcal{L}\{-t\} = [\mathcal{L}\{1\}]' = -\frac{1}{p^2}$. So

$$\mathcal{L}\{t\} - \mathcal{L}\left\{t - \alpha\left[\frac{t}{\alpha}\right]\right\} = \frac{\alpha}{p}\frac{e^{-\alpha p}}{1-e^{-\alpha p}},$$

and

$$\mathcal{L}^{-1}\left\{\frac{1}{p}\frac{e^{-\alpha p}}{1-e^{-\alpha p}}\right\} = \left[\frac{t}{\alpha}\right]\eta(t).$$

□

5. (3)

Proof. Denote by $F(p)$ the Laplace transform of $y(t)$. By the Convolution Theorem (Theorem 9.1), $F(p) = \frac{a}{p^2+1} - 2F(p)\frac{p}{p^2+1}$. So $F(p) = \frac{a}{(p+1)^2}$ and

$$y(t) = \mathcal{L}^{-1}\left\{\frac{a}{(p+1)^2}\right\} = -a\mathcal{L}^{-1}\left\{\left(\frac{1}{p+1}\right)'\right\} = ate^{-t}.$$

□

(4)

Proof. Denote by $F(p)$ the Laplace transform of $f(t)$. Then $F(p) + 2F(p)\frac{p}{p^2+1} = \frac{9}{p-2}$. Therefore

$$F(p) = \frac{9(p^2+1)}{(p-2)(p+1)^2} = -\frac{6}{(p+1)^2} + \frac{5}{p-2} + \frac{4}{p+1}.$$

Hence

$$f(t) = 6\mathcal{L}^{-1}\left\{\left(\frac{1}{p+1}\right)'\right\} + 5e^{2t} + 4e^{-t} = 5e^{2t} + 4e^{-t} - 6te^{-t}.$$

□

6. (1)

Proof. By the formula $\int_0^\infty F(p)dp = \int_0^\infty \frac{f(t)}{t}dt$, we have

$$\begin{aligned}\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\cos(cx)dx &= \int_0^\infty \mathcal{L}\{(e^{-ax} - e^{-bx})\cos(cx)\}dp \\ &= \int_0^p \left[\frac{a+p}{(a+p)^2+c^2} - \frac{b+p}{(b+p)^2+c^2}\right]dp \\ &= \frac{1}{2}\ln\frac{b^2+c^2}{a^2+c^2}.\end{aligned}$$

Here the Laplace transform inside the integral is obtained by applying integration-by-parts formula twice to integrals of the form $\int_0^\infty e^{-\omega x}\cos(cx)dx$. □

(2)

Proof. By the formula $\int_0^\infty F(p)dp = \int_0^\infty \frac{f(t)}{t}dt$ and the formula $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_p^\infty F(q)dq$, we have

$$\int_0^\infty \frac{1-\cos bx}{x^2}dx = \int_0^\infty \mathcal{L}\left\{\frac{1-\cos bx}{x}\right\}dp = \int_0^\infty\int_0^\infty \mathcal{L}\{1-\cos bx\}dqdp = \int_0^\infty\int_0^\infty \left(\frac{1}{q} - \frac{q}{q^2+b^2}\right)dqdp.$$

Note

$$\int_p^\infty \left(\frac{1}{q} - \frac{q}{q^2+b^2}\right)dq = \lim_{N\to\infty}\left(\ln\frac{N}{p} - \frac{1}{2}\ln\frac{N^2+b^2}{p^2+b^2}\right) = \frac{1}{2}\ln\frac{p^2+b^2}{p^2},$$

and

$$\begin{aligned}\int_0^\infty \frac{1}{2}\ln\frac{p^2+b^2}{p^2}dp &= \frac{1}{2}\left(p\ln\frac{p^2+b^2}{p^2}\bigg|_{p=0}^\infty - \int_0^\infty p\left(\frac{2p}{p^2+b^2} - \frac{2p}{p^2}\right)dp\right) \\ &= \int_0^\infty \frac{b^2}{p^2+b^2}dp = b\arctan\frac{p}{b}\bigg|_{p=0}^\infty = \frac{\pi}{2}b.\end{aligned}$$

□

8. (1)

Proof.

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{3n+1} = \sum_{n=0}^{\infty}(-1)^n\int_0^{\infty}e^{-t}e^{-3nt}dt = \int_0^{\infty}e^{-t}\sum_{n=0}^{\infty}(-e^{-3t})^n dt = \int_0^{\infty}\frac{e^{-t}}{1+e^{-3t}}dt.$$

Substituting e^{-t} for y, we have

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{(-1)^n}{3n+1} &= \int_0^1\frac{dy}{1+y^3} = \int_0^1\left[\frac{1}{3(y+1)} - \frac{1}{3}\frac{y-2}{y^2-y+1}\right]dy \\
&= \frac{1}{3}\int_0^1\left[\frac{1}{y+1} - \frac{y-\frac{1}{2}}{(y-\frac{1}{2})^2+\frac{3}{4}} + \frac{3}{2}\frac{1}{(y-\frac{1}{2})^2+\frac{3}{4}}\right]dy \\
&= \frac{1}{3}\left\{\ln(y+1) - \frac{1}{2}\ln\left[\left(y-\frac{1}{2}\right)^2+\frac{3}{4}\right] + \sqrt{3}\arctan\left(\frac{y-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)\right\}\Bigg|_{y=0}^{1} \\
&= \frac{1}{3}\left(\ln 2 + 2\sqrt{3}\arctan\frac{1}{\sqrt{3}}\right) \\
&= \frac{1}{3}\left(\ln 2 + \frac{\pi}{\sqrt{3}}\right).
\end{aligned}$$

□

(2)

Proof.

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{4n+1} = \sum_{n=0}^{\infty}(-1)^n\int_0^{\infty}e^{-(4n+1)t}dt = \int_0^{\infty}e^{-t}\sum_{n=0}^{\infty}(-e^{-4t})^n dt = \int_0^{\infty}\frac{e^{-t}}{1+e^{-4t}}dt.$$

Substituting e^{-t} for y, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(-1)^n}{4n+1} = \int_0^1\frac{dy}{1+y^4} = -\frac{1}{2\sqrt{2}}\int_0^1\left[\frac{y-\sqrt{2}}{y^2-\sqrt{2}y+1} - \frac{y+\sqrt{2}}{y^2+\sqrt{2}y+1}\right]dy \\
=\ & -\frac{1}{2\sqrt{2}}\left\{\frac{1}{2}\ln[(\sqrt{2}y-1)^2+1] - \arctan(\sqrt{2}y-1) - \frac{1}{2}\ln[(\sqrt{2}y+1)^2+1] - \arctan(\sqrt{2}y+1)\right\}\Bigg|_0^1 \\
=\ & -\frac{1}{2\sqrt{2}}\left\{\frac{1}{2}\ln\frac{2-\sqrt{2}}{2+\sqrt{2}} - \arctan(\sqrt{2}-1) - \arctan(\sqrt{2}+1)\right\} \\
=\ & -\frac{1}{2\sqrt{2}}\left[\frac{1}{2}\ln(3-2\sqrt{2}) - \frac{\pi}{2}\right] \\
=\ & \frac{1}{4\sqrt{2}}[2\ln(\sqrt{2}+1)+\pi].
\end{aligned}$$

□

(3)

Proof. Suppose $p, q \in \mathbb{N}$ and $q \leq p$, we have (substitute $e^{-\frac{t}{p}}$ for y)

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n+\frac{q}{p}} = \sum_{n=0}^{\infty}(-1)^n\int_0^{\infty}e^{-(n+q/p)t}dt = \int_0^{\infty}e^{-\frac{q}{p}t}\sum_{n=0}^{\infty}(-1)^n e^{-nt}dt = \int_0^{\infty}\frac{e^{-\frac{q}{p}t}}{1+e^{-t}}dt = \int_0^1\frac{py^{q-1}}{1+y^p}dy.$$

It's easy to verify (we have proved $\int_0^1 \frac{dy}{1+y^3} = \frac{1}{3}\left(\ln 2 + \frac{\pi}{\sqrt{3}}\right)$ in part (1) and note $\frac{y}{1+y^3} = -\frac{1}{3(y+1)} + \frac{y+1}{3(y^2-y+1)}$)

$$\int_0^1 \frac{py^{q-1}}{1+y^p}dy \begin{cases} \ln 2 & p=q=1; \\ \ln 2 + \frac{\pi}{\sqrt{3}} & p=3,\ q=1; \\ -\ln 2 + \frac{\pi}{\sqrt{3}} & p=3,\ q=2. \end{cases}$$

Therefore

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-1)^n}{(3n+1)(3n+2)(3n+3)} &= \sum_{n=0}^{\infty}(-1)^n \left[\frac{1}{6(n+1)} + \frac{1}{2(1+3n)} - \frac{1}{2+3n}\right] \\ &= \frac{1}{6}\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1} + \frac{1}{6}\sum_{n=0}^{\infty}\frac{(-1)^n}{n+\frac{1}{3}} - \frac{1}{3}\sum_{n=0}^{\infty}\frac{1}{n+\frac{2}{3}} \\ &= \frac{1}{6}\ln 2 + \frac{1}{6}\left(\ln 2 + \frac{\pi}{\sqrt{3}}\right) - \frac{1}{3}\left(-\ln 2 + \frac{\pi}{\sqrt{3}}\right) \\ &= \frac{2}{3}\ln 2 - \frac{\pi}{6\sqrt{3}}. \end{aligned}$$

□

10 δ Function

1. Let $\varphi(x)$ be any test function that satisfies certain regularity conditions.

(1)

Proof. $\int_{-\infty}^{\infty} \varphi(x)\delta(-x)dx = \int_{-\infty}^{\infty} \varphi(-x)\delta(x)dx = \varphi(0) = \int_{-\infty}^{\infty} \varphi(x)\delta(x)dx$. So $\delta(-x) = \delta(x)$. □

(2)

Proof. $\int_{-\infty}^{\infty} \varphi(x) \cdot x\delta(x)dx = \int_{-\infty}^{\infty} (x\varphi(x))\delta(x)dx = 0 \cdot \varphi(0) = 0$. So $x\delta(x) = 0$. □

(3)

Proof. $\int_{-\infty}^{\infty} \varphi(x) \cdot f(x)\delta(x)dx = \varphi(0)f(0) = \int_{-\infty}^{\infty} \varphi(x) \cdot f(0)\delta(x)dx$. So $f(x)\delta(x) = f(0)\delta(x)$. □

(4)

Proof.

$$\begin{aligned} \int_{-\infty}^{\infty} \varphi(x) \cdot x\delta'(x)dx &= \int_{-\infty}^{\infty} (x\varphi(x))\delta'(x)dx = x\varphi(x)\delta(x)|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} [x\varphi(x)]'\delta(x)dx \\ &= -\varphi(0) = -\int_{-\infty}^{\infty} \varphi(x)\delta(x)dx. \end{aligned}$$

So $x\delta'(x) = -\delta(x)$. □

(5)

Proof. $\int_{-\infty}^{\infty} \varphi(x)\delta(ax)dx = \int_{-\infty}^{\infty} \varphi(y/a)\delta(y)\frac{dy}{a} = \frac{1}{a}\varphi(0)$. So $\delta(ax) = \frac{1}{a}\delta(x)$. □

(6)

Proof.

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\delta(x^2-a^2)dx &= \int_{-\infty}^{0} f(x)\delta(x^2-a^2)dx + \int_{0}^{\infty} f(x)\delta(x^2-a^2)dx \\ &= \int_{\infty}^{-a^2} f(-\sqrt{y+a^2})\delta(y)d(-\sqrt{y+a^2}) + \int_{-a^2}^{\infty} f(\sqrt{y+a^2})\delta(y)d\sqrt{y+a^2} \\ &= \int_{-a^2}^{\infty} \frac{f(-\sqrt{y+a^2})}{2\sqrt{y+a^2}}\delta(y)dy + \int_{-a^2}^{\infty} \frac{f(\sqrt{y+a^2})}{2\sqrt{y+a^2}}\delta(y)dy \\ &= \frac{1}{2a}[f(-a)+f(a)],\end{aligned}$$

where the last equality is due to the fact $0 \in (-a^2, \infty)$. Therefore, $\delta(x^2-a^2) = \frac{1}{2a}[\delta(x-a)+\delta(x+a)]$. □

Remark 14. *More generally, we have the following useful result*

Proposition 1. *Suppose $\varphi(x)$ is a continuously differentiable function and the equation $\varphi(x) = 0$ has finitely many roots $(x_k)_{k=1}^N$ with $\varphi(x_k) \neq 0$. Then*

$$\delta[\varphi(x)] = \sum_{k=1}^{N} \frac{\delta(x-x_k)}{|\varphi'(x_k)|}.$$

Proof. For each $k \in \{1, \cdots, N\}$, we prove $\delta(\varphi(x))$ is $C_k\delta(x-x_k)$ for some constant C_k in a neighborhood of x_k. Indeed, we can find $\varepsilon > 0$ such that $x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_N \notin [x_k-\varepsilon, x_k+\varepsilon]$. Clearly $\delta(\varphi(x_k)) = \infty$. Furthermore, we have by the change-of-variable formula ($y := \varphi(x)$)

$$\int_{x_k-\varepsilon}^{x_k+\varepsilon} \delta(\varphi(x))dx = \int_{[\varphi(x_k-\varepsilon),\varphi(x_k+\varepsilon]} \frac{\delta(y)dy}{|\varphi'(\varphi^{-1}(y))|} = \frac{1}{|\varphi'(\varphi^{-1}(\varphi(x_k)))|} = \frac{1}{|\varphi'(x_k)|}.$$

So in a sufficiently small neighborhood of x_k, $\delta(\varphi(x)) = \frac{\delta(x-x_k)}{|\varphi'(x_k)|}$. □

2. (1)

Proof. The general solution of the homogeneous equation $\left[\frac{d^2}{dx^2} - k^2\right] g(x;t) = 0$ $(x > t)$ is $c_1(t)e^{kx}+c_2(t)e^{-kx}$. By the continuity property of $g(x;t)$ at $x = t$ (formula (10.36a) and formula (10.36b)), we have

$$\begin{cases} c_1(t)e^{kt} + c_2(t)e^{-kt} = 0 \\ kc_1(t)e^{kt} - kc_2(t)e^{-kt} = 1. \end{cases}$$

Solving this equation, we get $c_1(t) = \frac{e^{-kt}}{2k}$ and $c_2(t) = -\frac{e^{kt}}{2k}$. Therefore, combining with formula (10.39), we have

$$g(x;t) = [c_1(t)e^{kx} + c_2(t)e^{-kx}]\eta(x-t) = \frac{1}{k}\sinh k(x-t)\eta(x-t).$$

Remark 15. *The above result differs from the answer of the textbook. But according to the textbook's answer to Exercise Problem 3(2) of this chapter, we see the correct answer is indeed $\frac{1}{k}\sinh k(x-t)\eta(x-t)$.*

□

(2)

Proof. By Exercise Problem 2(1) of Chapter 6, the general solution of the homogenous equation $\left[\frac{d^2}{dx^2} - x^2\right] g(x;t) = \delta(x-t)$ $(x > t)$ is $c_1(t)w_1(x) + c_2(t)w_2(x)$ where

$$w_1(x) = \sum_{n=0}^{\infty} \frac{\Gamma(3/4)}{n!\Gamma(n+3/4)} \left(\frac{x}{2}\right)^{4n}, \quad w_2(x) = \sum_{n=0}^{\infty} \frac{\Gamma(5/4)}{n!\Gamma(n+5/4)} \left(\frac{x}{2}\right)^{4n+1}.$$

By the continuity property of $g(x;t)$ at $x=t$ (formula (10.36a) and formula (10.36b)), we have

$$\begin{cases} c_1(t)w_1(t)+c_2(t)w_2(t)=0 \\ c_1(t)w_1'(t)+c_2(t)w_2'(t)=1. \end{cases}$$

Using the hint that $\begin{vmatrix} w_1(x) & w_2(x) \\ w_1'(x) & w_2'(x) \end{vmatrix} = \frac{1}{2}$, we can solve the above equations to get $c_1(t)=-2w_2(t)$ and $c_2(t)=2w_1(t)$. Therefore, combining with formula (10.39), we have

$$g(x;t)=2[w_2(x)w_1(t)-w_1(x)w_2(t)]\eta(x-t).$$

Remark 16. *To see why the hint is true, we note* $[w_1(x)w_2'(x)-w_1'(x)w_2(x)]'=w_1(x)w_2''(x)-w_1''(x)w_2(x)=w_1(x)\cdot x^2w_2(x)-x^2w_1(x)w_2(x)=0$. *Therefore,* $w_1(x)w_2'(x)-w_1'(x)w_2(x)=const=w_1(0)w_2'(0)-w_1'(0)w_2(0)=1/2$.

□

(3)

Proof. By Exercise Problem 2(4) of Chapter 6, the homogenous equation

$$[(1+x+x^2)\frac{d^2}{dx^2}+2(1+2x)\frac{d}{dx}+2]g(x;t)=0$$

has solution $c_1(t)w_1(x)+c_2(t)w_2(x)$, where

$$w_1(x)=\frac{1}{1+x+x^2},\ w_2(x)=\frac{x}{1+x+x^2}.$$

To use the continuity conditions of the Green's function at $x=t$, we note

$$\left[(1+x+x^2)\frac{d^2}{dx^2}+2(1+2x)\frac{d}{dx}+2\right]g(x;t)=\frac{d^2}{dx^2}[(1+x+x^2)g(x;t)].$$

So by the condition $\left.\frac{dg(x;t)}{dx}\right|_{x<t}=0$ and integrating both sides of the equation, we have

$$\frac{d}{dx}[(1+x+x^2)g(x;t)]|_{t-0}^{t+0}=1,$$

i.e. $(1+2t)[c_1(t)w_1(t)+c_2(t)w_2(t)]+(1+t+t^2)[c_1(t)w_1'(t)+c_2(t)w_2'(t)]=1$. And by the continuity of $g(x;t)$ at $x=t$ and $g(x;t)|_{x<t}=0$, we have $c_1(t)w_1(t)+c_2(t)w_2(t)=0$. Combined, we have the system of equations

$$\begin{cases} c_1(t)w_1(t)+c_2(t)w_2(t)=0 \\ c_1(t)w_1'(t)+c_2(t)w_2'(t)=\frac{1}{1+t+t^2}. \end{cases}$$

Let $D(t)=\begin{vmatrix} w_1(t) & w_2(t) \\ w_1'(t) & w_2'(t) \end{vmatrix}$. Then it's easy to see $D(t)=\frac{1}{(1+t+t^2)^2}$. Then

$$\begin{bmatrix} c_1(t) \\ c_2(t) \end{bmatrix}=\begin{bmatrix} w_1(t) & w_2(t) \\ w_1'(t) & w_2'(t) \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ \frac{1}{1+t+t^2} \end{bmatrix}=D(t)^{-1}\begin{bmatrix} w_2'(t) & -w_2(t) \\ -w_1'(t) & w_1(t) \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ \frac{1}{1+t+t^2} \end{bmatrix}=\begin{bmatrix} -t \\ 1 \end{bmatrix}.$$

So $g(x;t)=\frac{x-t}{1+x+x^2}\eta(x-t)$. □

3. (1)

Proof. By Example 10.5, the solution to the homogeneous equation

$$\begin{cases} \frac{d^2g(x;t)}{dx^2} + k^2 g(x;t) = \delta(x-t), \ x, t > 0 \\ g(0;t) = 0, \ \frac{dg(x;t)}{dx}\Big|_{x=0} = 0. \end{cases}$$

is $g(x;t) = \frac{1}{k}\sin k(x-t)\eta(x-t)$. By formula (10.60),

$$\begin{aligned} y(x) &= \int_0^x g(x;t)f(t)dt - \left[A\frac{dg(x;t)}{dt} - Bg(x;t)\right]_{t=0} \\ &= \int_0^x \frac{1}{k}\sin k(x-t)f(t)dt - \left[-A\cos k(x-t) - \frac{B}{k}\sin k(x-t)\right]_{t=0} \\ &= \frac{1}{k}\int_0^x \sin k(x-t)f(t)dt + A\cos kx + \frac{B}{k}\sin kx. \end{aligned}$$

□

(2)

Proof. By Exercise Problem 2(1) of this chapter, the homogeneous equation

$$\begin{cases} \frac{d^2g(x;t)}{dx^2} - k^2 g(x;t) = \delta(x-t), \ x, t > 0 \\ g(0;t) = 0, \ \frac{dg(x;t)}{dx}\Big|_{x=0} = 0. \end{cases}$$

has solution $g(x;t) = \frac{1}{k}\sinh k(x-t)\eta(x-t)$. By formula (10.60), we have

$$y(x) = \int_0^x g(x;t)f(t)dt - \left[A\frac{dg(x;t)}{dt} - Bg(x;t)\right]_{t=0} = A\cosh kx + \frac{B}{k}\sinh kx + \frac{1}{k}\int_0^x \sinh k(x-t)f(t)dt.$$

□

(3)

Proof. By Exercise Problem 2(2) of this chapter, the Green's function is

$$g(x;t) = 2[w_2(x)w_1(t) - w_1(x)w_2(t)]\eta(x-t),$$

where $w_1(x) = \sum_{n=0}^{\infty} \frac{\Gamma(3/4)}{n!\Gamma(n+3/4)}\left(\frac{x}{2}\right)^{4n}$ and $w_2(x) = \sum_{n=0}^{\infty} \frac{\Gamma(5/4)}{n!\Gamma(n+5/4)}\left(\frac{x}{2}\right)^{4n+1}$. Define

$$D_1(t,x) = \begin{vmatrix} w_1(t) & w_2(t) \\ w_1(x) & w_2(x) \end{vmatrix}, \ D_2(t,x) = \begin{vmatrix} w_1(t) & w_2(t) \\ w_1'(x) & w_2'(x) \end{vmatrix}.$$

Then $g(x;t) = 2D_1(t,x)\eta(x-t)$ and by formula (10.60)

$$\begin{aligned} y(x) &= \int_0^x g(x;t)f(t)dt - \left[A\frac{dg(x;t)}{dt} - Bg(x;t)\right]_{t=0} \\ &= 2\int_0^x D_1(t,x)f(t)dt - A\cdot 2[w_2(x)w_1'(0) - w_1(x)w_2'(0)] + B\cdot 2[w_2(x)w_1(0) - w_1(x)w_2(0)] \\ &= 2\int_0^x D_1(t,x)f(t)dt + 2AD_2(x,0) + 2BD_1(0,x). \end{aligned}$$

Remark 17. *This result differs from the answer of the textbook. Check.*

□

4. (1)

Proof. By Exercise Problem 2(1) of this chapter, the general solution to the equation

$$\left[\frac{d^2}{dx^2} - k^2\right] g(x;t) = \delta(x-t)$$

is $\frac{1}{k}\sinh k(x-t)\eta(x-t) + C(t)e^{kx} + D(t)e^{-kx}$. Plugging this formula into the boundary conditions, we get

$$\begin{cases} C(t) + D(t) = 0 \\ \frac{1}{k}\sinh k(1-t) + C(t)e^{k} + D(t)e^{-k} = 0. \end{cases}$$

Solving it gives us $D(t) = \frac{1}{2k}\frac{\sinh k(1-t)}{\sinh k}$ and $C(t) = -\frac{1}{2k}\frac{\sinh k(1-t)}{\sinh k}$. Therefore

$$g(x;t) = \frac{1}{k}\sinh k(x-t)\eta(x-t) - \frac{\sinh k(1-t)}{k\sinh k}\sinh kx.$$

□

(2)

Proof. By Exercise Problem 2(2) of this chapter, the general solution to the equation

$$\left[\frac{d^2}{dx^2} - x^2\right] g(x;t) = \delta(x-t)$$

is $2[w_2(x)w_1(t) - w_1(x)w_2(t)]\eta(x-t) + C(t)w_1(x) + D(t)w_2(t)$, where

$$w_1(x) = \sum_{n=0}^{\infty} \frac{\Gamma(3/4)}{n!\Gamma(n+3/4)}\left(\frac{x}{2}\right)^{4n},\ w_2(x) = \sum_{n=0}^{\infty} \frac{\Gamma(5/4)}{n!\Gamma(n+5/4)}\left(\frac{x}{2}\right)^{4n+1}.$$

Plugging this formula into the boundary conditions, we get (note $w_1(0) = 1$ and $w_2(0) = 0$)

$$\begin{cases} C(t) = 0 \\ 2[w_2(1)w_1(t) - w_1(1)w_2(t)] + C(t)w_1(1) + D(t)w_2(1) = 0. \end{cases}$$

Solving it gives us $C(t) = 0$ and $D(t) = -\frac{2D_1(t,1)}{w_2(1)}$. So

$$g(x;t) = -\frac{2w_2(x)}{w_2(1)}D_1(t,1) + 2D_1(t,x)\eta(x-t),$$

where $D_1(t,x) = \begin{vmatrix} w_1(t) & w_2(t) \\ w_1(x) & w_2(x) \end{vmatrix}$.

Remark 18. *The above result differs from the answer in the textbook. But according to the textbook's answer to Exercise Problem 5(3), the above result is the correct one.*

□

(3)

Proof. By Exercise Problem 2(3) of this chapter, the general solution to the quation

$$\left[(1+x+x^2)\frac{d^2}{dx^2} + 2(1+2x)\frac{d}{dx} + 2\right] g(x;t) = \delta(x-t)$$

is $\frac{x-t}{1+x+x^2}\eta(x-t) + \frac{C(t)}{1+x+x^2} + \frac{D(t)}{1+x+x^2}x$. Plugging this formula into the boundary conditions, we get $C(t) = 0$ and $D(t) = -(l-t)/l$. So

$$g(x;t) = \frac{x-t}{1+x+x^2}\eta(x-t) - \frac{(l-t)x}{l(1+x+x^2)}.$$

Remark 19. *The textbook's answer is wrong, as seen easily by checking the boundary condition at $x = l$.*

□

5. (1)

Proof. By Example 10.7, the solution to the equation

$$\begin{cases} \left[\frac{d^2}{dx^2} + k^2\right] g(x;t) = \delta(x-t) \quad (0 < x, t < 1) \\ g(0;t) = 0, \ g(1;t) = 0 \end{cases}$$

is $g(x;t) = \frac{1}{k}\sin k(x-t)\eta(x-t) - \frac{1}{k}\frac{\sin k(1-t)}{\sin k}\sin kx$. By formula (10.67), we have

$$\begin{aligned} y(x) &= \int_0^1 g(x;t)f(t)dt + B\frac{\cos k(1-t)}{\sin k}\sin kx\bigg|_{t=1} - A\left[-\cos k(x-t) + \frac{\cos k(1-t)}{\sin k}\sin kx\right]_{t=0} \\ &= \frac{1}{k}\int_0^x \sin k(x-t)f(t)dt - \frac{\sin kx}{k\sin k}\int_0^1 \sin k(1-t)f(t)dt + B\frac{\sin kx}{\sin k} + A\frac{\sin k(1-x)}{\sin k}. \end{aligned}$$

□

(2)

Proof. By Exercise Problem 4(1) of this chapter, the solution to the equation

$$\begin{cases} \left[\frac{d^2}{dx^2} - k^2\right] g(x;t) = \delta(x-t) \quad (0 < x, t < 1, k > 0) \\ g(0;t) = 0, \ g(1,t) = 0. \end{cases}$$

is $g(x;t) = \frac{1}{k}\sinh k(x-t)\eta(x-t) - \frac{\sinh k(1-t)}{k\sinh k}\sinh kx$. Using formula (10.6), we have

$$\begin{aligned} y(x) &= \frac{1}{k}\int_0^x \sinh k(x-t)f(t)dt - \frac{\sinh kx}{k\sinh k}\int_0^1 \sinh k(1-t)f(t)dt + B\frac{\cosh k(1-t)}{\sinh k}\sinh kx\bigg|_{t=1} \\ &\quad -A\left[\cosh k(x-t) + \frac{\cosh k(1-t)}{\sinh k}\sinh kx\right]_{t=0} \\ &= \frac{1}{k}\int_0^x \sinh k(x-t)f(t)dt - \frac{\sinh kx}{k\sinh k}\int_0^1 \sinh k(1-t)f(t)dt + B\frac{\sinh kx}{\sinh k} - A\frac{\sinh k(x+1)}{\sinh k}. \end{aligned}$$

Remark 20. *The above result differs from the textbook's answer. Check.*

□

(3)

Proof. The solution to the equation

$$\begin{cases} \left[\frac{d^2}{dx^2} - x^2\right] g(x;t) = \delta(x-t) \quad (0 < x, t < 1) \\ g(0;t) = 0, \ g(1;t) = 0 \end{cases}$$

is $g(x;t) = -\frac{2w_2(x)}{w_2(1)}D_1(t,1) + 2D_1(t,x)\eta(x-t)$. By formula (10.67)

$$\begin{aligned} y(x) &= \int_0^1 g(x;t)f(t)dt + B\left.\frac{dg(x;t)}{dt}\right|_{t=1} - A\left.\frac{dg(x;t)}{dt}\right|_{t=0} \\ &= 2\int_0^x D_1(t,x)f(t)dt - \frac{2w_2(x)}{w_2(1)}\int_0^1 D_1(t,1)f(t)dt - B\frac{2w_2(x)}{w_2(1)}[w_1'(1)w_2(1) - w_2'(1)w_1(1)] \\ &\quad -A\left\{-\frac{2w_2(x)}{w_2(1)}[w_1'(0)w_2(1) - w_2'(0)w_1(1)] + 2[w_1'(0)w_2(x) - w_2'(0)w_1(x)]\right\} \\ &= 2\int_0^x D_1(t,x)f(t)dt - \frac{2w_2(x)}{w_2(1)}\int_0^1 D_1(t,1)f(t)dt + B\frac{2w_2(x)}{w_2(1)}D_2'(1,1) \\ &\quad -A\left[\frac{w_1(1)}{w_2(1)}w_2(x) - w_1(x)\right]. \end{aligned}$$

Remark 21. *The above result differs from the textbook's answer. Check.*

□

11 Complex Functions in Mathematica

There are no exercise problems for this chapter.

12 Equations of Mathematical Physics

1.

Proof. For the points inside the bar, we can apply the partial differential equation (12.10). For the boundary conditions, the end where $x = 0$ is fixed, so $u(0,t) = 0$; the end where $x = l$ has no external stress, so by Hooke's law (formula (12.9)) $\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=l} = 0$ (see formula (12.36)). For the initial conditions, the initial velocity of every point on the bar is 0, so $\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = 0$; at time 0, Hooke's law implies $E\frac{u(x,0)}{x} = P = \frac{F}{S}$, so $u|_{t=0} = \frac{F}{ES}x$. Combined, we have

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = 0, & \\ u|_{x=0} = 0, & \left.\frac{\partial u}{\partial x}\right|_{x=l} = 0, \\ u|_{t=0} = \frac{F}{ES}x, & \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0. \end{cases}$$

□

2.

Proof. Let D be the rate of diffusion. Then from formula (12.20), we conclude

$$\frac{\partial u}{\partial t} = D\bigtriangledown^2 u + \alpha u.$$

□

3.

Proof. By Fourier's law (formula (12.15)), we have

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = -\frac{q_1}{k},\ \left.\frac{\partial u}{\partial x}\right|_{x=l} = \frac{q_2}{k}.$$

□

4.

Proof. We choose polar coordinate and place the origin of the coordinate at the center of the ball, with axis pointing to the sun. Then by formula (12.41), the boundary conditions are

$$\left[\frac{\partial u}{\partial r} + \frac{H}{k}u\right]_{r=a} = \frac{H}{k}u_0 = \begin{cases} \frac{M}{k}\cos\theta, & 0 \le \theta \le \frac{\pi}{2}, \\ 0, & \frac{\pi}{2} < \theta \le \pi, \end{cases}$$

where H is the proportional constant in Newton's law of cooling. □

13 General Solutions of Linear PDE

13.1 Exercise in the text

13.1.

Proof. Suppose the solution has the form $u(x, y) = g(y)\phi(y + \alpha x)$. Then

$$(D_x - \alpha D_y - \beta)u = g(y)(D_x - \alpha D_y)\phi(y + \alpha x) + \phi(y + \alpha x)(-\alpha D_y - \beta)g(y).$$

Since $(D_x - \alpha D_y)\phi(y + \alpha x) = 0$, we obtain the ODE for $g(y)$: $(\alpha D_y + \beta)g(y) = 0$, which has a solution $g(y) = e^{-\frac{\beta}{\alpha}y}$. So $u(x, y) = e^{-\frac{\beta}{\alpha}}\phi(y + \alpha x)$. □

13.2 Exercise at the end of chapter

1. (1)

Proof. The auxiliary equation is $\alpha^2 - 2\alpha - 3 = 0$, which has roots 3 and -1. So the general solution has the form of $f(3x + y) + g(x - y)$, where f and g are two independent C^2 (twice differentiable) functions. □

(2)

Proof. The auxiliary equation is $\alpha^2 - 2\alpha + 2 = 0$, which has roots $1 \pm i$. So the general solution has the form of $f(x + y + ix) + g(x + y - ix)$, where f and g are two independent C^2 functions. □

(3)

Proof. The auxiliary equation is $\alpha^2 - \alpha = 0$. So the general solution has the form of $f(y) + g(y + x)$, where f and g are two independent C^2 functions. □

(4)

Proof. We consider the PDE for $ru(t, r)$. The original PDE gives us $D_t^2 u = \frac{c^2}{r^2}(2rD_r u + r^2 D_r^2 u)$, which is equivalent to

$$[D_t^2 - c^2 D_r^2](ru) = rD_t^2 u - 2c^2 D_r u - c^2 r D_r^2 u = 0.$$

So the auxiliary equation for $ru(t, r)$ is $\alpha^2 - c^2 = 0$, which has roots $\pm c$. So $ru(t, r)$ has the general form of $f(r + ct) + g(r - ct)$. Therefore $u(t, r)$ has the general form of $\frac{1}{r}[f(r + ct) + g(r - ct)]$. □

(5)

Proof. The auxiliary equation is $(a^2 - b^2)\alpha^2 + 2a\alpha + 1 = 0$, which has roots $-\frac{1}{a+b}$ and $-\frac{1}{a-b}$. So the general solution has the form of $f\left(t - \frac{1}{a-b}x\right) + g\left(t - \frac{1}{a+b}x\right)$, or equivalently, $f(x - (a + b)t) + g(x - (a - b)t)$, where f and g are two independent C^2 functions. □

(6)

Proof. The auxiliary equation is $\alpha^4 - 1 = 0$, which has roots $\pm i$, ± 1. So the general solution has the form of $\phi_1(y+x) + \phi_2(y-x) + \phi_3(y+ix) + \phi_4(y-ix)$. □

2. (1)

Proof. The general solution to the homogeneous equation

$$\frac{\partial^2 u}{\partial x} + \frac{\partial^2 u}{\partial y^2} = 0$$

has the form of $f(x+iy) + g(x-iy)$ where f and g are linearly independent (formula (13.11)). To find a special solution, we note

$$\frac{1}{D_x^2 + D_y^2}(x^2+xy) = \frac{1}{D_x^2}\left[\sum_{n=0}^{\infty}(-1)^n\left(\frac{D_y^2}{D_x^2}\right)^n\right](x^2+xy) = \frac{1}{D_x^2}(x^2+xy) = \frac{x^4}{12} + \frac{x^3 y}{6}.$$

So the general solution to the original equation has the form of

$$f(x+iy) + g(x-iy) + \frac{x^4}{12} + \frac{x^3 y}{6}.$$

□

(2)

Proof. The general solution to the homogeneous equation

$$\frac{\partial^2 u}{\partial x} - \frac{\partial^2 u}{\partial y^2} = 0$$

has the form of $f(x+y) + g(x-y)$ where f and g are linearly independent (formula (13.11)). To find a special solution, we note

$$\frac{1}{D_x^2 - D_y^2}(xy-x) = \frac{1}{D_x^2}\left[\sum_{n=0}^{\infty}\left(\frac{D_y^2}{D_x^2}\right)^n\right](xy-x) = \frac{1}{D_x^2}(xy-x) = \frac{x^3 y}{6} - \frac{x^3}{6}.$$

So the general solution to the original equation has the form of

$$f(x+y) + g(x-y) + \frac{1}{6}x^3(y-1).$$

Remark 22. *The textbook's answer is* $f(x+y)+g(x-y)+\frac{1}{6}x^3(y+1)$*, which can be easily verified as wrong.*

□

(3)

Proof. The auxiliary equation of the homogeneous equation

$$\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0$$

is $\alpha^2 - 2\alpha + 1 = 0$. So the general solution to the homogeneous equation has the form of $x\phi(x+y) + \psi(x+y)$, where ϕ and ψ are linearly independent functions. To find a special solution, note

$$\begin{aligned}\frac{1}{(D_x - D_y)^2}(x^2+y) &= \frac{1}{D_x^2}\left[\sum_{n=0}^{\infty}\left(\frac{D_y}{D_x}\right)^n\right]^2(x^2+y) = \frac{1}{D_x^2}\left(1 + \frac{D_y}{D_x} + \cdots\right)^2(x^2+y) \\ &= \frac{1}{D_x^2}\left(1 + \frac{2D_y}{D_x}\right)(x^2+y) = \frac{1}{D_x^2}(x^2 + y + \frac{2}{D_x}1) = \frac{x^4}{12} + \frac{x^2 y}{2} + \frac{x^3}{3}.\end{aligned}$$

So the general solution to the original equation has the form of

$$x\phi(x+y) + \psi(x+y) + \frac{x^4}{12} + \frac{x^3}{3} + \frac{x^2 y}{2}.$$

Remark 23. *The above result is different from the textbook's answer. Check.*

□

3. (1)

Proof. Using the transformation $x = e^t$ and $y = e^s$, we have

$$\begin{aligned} & x^2\frac{\partial^2 u}{\partial x^2} - 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} \\ = \;& D_t(D_t-1) - 2D_tD_s + D_s(D_s-1) + D_t + D_s \\ = \;& (D_t - D_s)^2. \end{aligned}$$

So the general solution has the form

$$u(x,y) = t\phi(t+s) + \psi(t+s) = \ln x\phi(\ln x + \ln y) + \psi(\ln x + \ln y) = \ln x \cdot f(xy) + g(xy),$$

where ϕ and ψ (or equivalently, f and g) are linearly independent functions. □

(2)

Proof. It's easy to see $\sin(xy)$ is a special solution to the inhomogeneous equation and $f(x+y)+g(x-y)$ is the generals solution to the corresponding homogeneous equation, where f and g are linearly independent functions. So the general solution to the inhomogeneous equation is $f(x+y)+g(x-y)+\sin(xy)$. □

4.

Proof. The key is to note that

$$\begin{aligned} & \frac{\partial}{\partial x}\left[\left(1-\frac{x}{l}\right)^2\frac{\partial u}{\partial x}\right] - \frac{1}{a^2}\left(1-\frac{x}{l}\right)^2\frac{\partial^2 u}{\partial t^2} \\ = \;& \frac{1}{l^2}\left[-2(l-x)\frac{\partial u}{\partial x} + (l-x)^2\frac{\partial^2 u}{\partial x^2} - \frac{1}{a^2}(l-x)^2\frac{\partial^2 u}{\partial t^2}\right] \\ = \;& \frac{l-x}{l^2}\left\{\frac{\partial^2}{\partial x^2}[(l-x)u] - \frac{1}{a^2}\frac{\partial^2}{\partial t^2}[(l-x)u]\right\} \end{aligned}$$

Define $v(x,t) = (l-x)u(x,t)$, we can get a new system of equations

$$\begin{cases} \frac{\partial^2}{\partial x^2}v(x,t) - \frac{1}{a^2}\frac{\partial^2}{\partial t^2}v(x,t) = 0 \\ v|_{t=0} = (l-x)\phi(x), \;\; \frac{\partial v}{\partial t}\Big|_{t=0} = (l-x)\psi(x). \end{cases}$$

By Example 13.9, we conclude

$$v(x,t) = \frac{1}{2}[(l-x+at)\phi(x-at) + (l-x-at)\phi(x+at)] + \frac{1}{2a}\int_{x-at}^{x+at}(l-\xi)\phi(\xi)d\xi.$$

So

$$u(x,t) = \frac{1}{2(l-x)}[(l-x+at)\phi(x-at) + (l-x-at)\phi(x+at)] + \frac{1}{2a(l-x)}\int_{x-at}^{x+at}(l-\xi)\phi(\xi)d\xi.$$

Remark 24. *The above result is different from the textbook's answer. Check.*

□

14 Separation of Variables

14.1 Exercise in the text

14.1.

Proof. If we solve for $T(t)$ directly, we can conclude the general solution to the equation $T''(t)+\lambda a^2 T(t)=0$ is $T(t)=A\sin(\sqrt{\lambda}at)+B\cos(\sqrt{\lambda}at)$. Boundary conditions give $A=B=0$. So $T(t)\equiv 0$.

If we apply theory of ordinary differential equations, we note by Theorem 6.1,

$$\begin{cases} T''(t)+\lambda a^2 T(t)=0 \\ T(0)=0,\ T'(0)=0 \end{cases}$$

has a unique solution in $(0,\infty)$, which has to be 0. □

14.2.

Proof. Assume u is C^2, then

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}-a^2\frac{\partial^2 u}{\partial x^2}=0, & 0<x<l, t>0, \\ \frac{\partial u(x,t)}{\partial x}\Big|_{x=0}=0,\ \frac{\partial u(x,t)}{\partial x}\Big|_{x=l}=0, & t\geq 0, \\ u|_{t=0}=\phi(x),\ \frac{\partial u}{\partial t}\Big|_{t=0}=\psi(x), & 0\leq x\leq l \end{cases}$$

implies

$$\phi'(0)=\lim_{x\to 0}\frac{\partial}{\partial x}(u|_{t=0})=\lim_{t\to 0}\frac{\partial u(x,t)}{\partial x}\Big|_{x=0}=0,\ \phi'(l)=\lim_{x\to l}\frac{\partial}{\partial x}(u|_{t=0})=\lim_{t\to 0}\frac{\partial u(x,t)}{\partial x}\Big|_{x=l}=0,$$

and

$$\psi'(0)=\lim_{x\to 0}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\Big|_{t=0}\right)=\lim_{t\to 0}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\Big|_{x=0}\right)=0,\ \psi'(l)=\lim_{x\to l}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\Big|_{t=0}\right)=\lim_{t\to 0}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\Big|_{x=l}\right)=0.$$

We should extend $\phi(x)$ and $\psi(x)$ in such a way that the extended functions are at least C^1. So

$$\Phi(x)=\begin{cases}\phi(-x), & -l\leq x\leq 0, \\ \phi(x), & 0\leq x\leq l,\end{cases}\quad \Psi(x)=\begin{cases}\psi(-x), & -l\leq x\leq 0, \\ \psi(x), & 0\leq x\leq l,\end{cases}$$

and then extend $\Phi(x)$ and $\Psi(x)$ to $(-\infty,\infty)$ as periodic functions with period $2l$. □

14.3.

Proof. Assume u is C^2, then similar to previous exercise problem,

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}-a^2\frac{\partial^2 u}{\partial x^2}=0, & 0<x<l, t>0, \\ u(x,t)|_{x=0}=0,\ \frac{\partial u(x,t)}{\partial x}\Big|_{x=l}=0, & t\geq 0, \\ u|_{t=0}=\phi(x),\ \frac{\partial u}{\partial t}\Big|_{t=0}=\psi(x), & 0\leq x\leq l \end{cases}$$

implies

$$\phi(0)=0,\ \phi'(l)=0,\ \psi(0)=0,\ \psi'(l)=0.$$

We should extend $\phi(x)$ and $\psi(x)$ in such a way that the extended functions are at least C^1. Therefore, we should first extend $\phi(x)$ as

$$\Phi(x)=\begin{cases}\phi(2l-x), & l\leq x\leq 2l, \\ \phi(x), & 0\leq x\leq l;\end{cases}$$

then we extend $\Phi(x)$ to $[-2l,2l]$ so that it becomes an odd function; finally, we extend $\Phi(x)$ to $(-\infty,\infty)$ as a periodic function with period $4l$. $\psi(x)$ should be extended similarly. □

14.4.

Proof. If we take choice (1), then the general solution to equation (14.21) will have the form

$$u(x,y)=\sum_{n=0}^{\infty}\left[A_n\exp\left\{\frac{2n+1}{2a}\pi y\right\}+B_n\exp\left\{-\frac{2n+1}{2a}\pi y\right\}\right]\sin\frac{2n+1}{2a}\pi x.$$

Plug this formula into (14.21c), we get

$$\begin{cases}\sum_{n=0}^{\infty}(A_n+B_n)\sin\frac{2n+1}{2a}\pi x=f(x)\\ \sum_{n=0}^{\infty}\frac{2n+1}{2a}\pi\left[A_n\exp\left\{\frac{2n+1}{2a}\pi b\right\}-B_n\exp\left\{-\frac{2n+1}{2a}\pi b\right\}\right]\sin\frac{2n+1}{2a}\pi x=0.\end{cases}$$

Using the orthogonality of the system $\left(\sin\frac{2n+1}{2a}\pi x\right)_{n=0}^{\infty}$ over $[0,a]$, we have

$$\begin{cases}A_n+B_n=\frac{2}{a}\int_0^a f(x)\sin\frac{2n+1}{2a}\pi x dx\\ A_n\exp\left\{\frac{2n+1}{2a}\pi b\right\}-B_n\exp\left\{-\frac{2n+1}{2a}\pi b\right\}=0.\end{cases}$$

Solving the equations gives us

$$B_n=\frac{\frac{2}{a}\int_0^a f(x)\sin\frac{2n+1}{2a}\pi x dx}{1+\exp\left\{-\frac{2n+1}{a}\pi b\right\}},\ A_n=\frac{\exp\left\{-\frac{2n+1}{a}\pi b\right\}\frac{2}{a}\int_0^a f(x)\sin\frac{2n+1}{2a}\pi x dx}{1+\exp\left\{-\frac{2n+1}{a}\pi b\right\}}.$$

The result is the same as choosing the following form for $Y_n(y)$:

$$Y_n(y)=C_n\sinh\frac{2n+1}{2a}\pi y+D_n\cosh\frac{2n+1}{2a}\pi y.$$

But clearly choice (1) makes the result look messier.

If we take choice (2), the general solution to equation (14.21) will have the form

$$u(x,y)=\sum_{n=0}^{\infty}\left[A_n\sinh\frac{2n+1}{2a}\pi y+B_n\cosh\frac{2n+1}{2a}\pi(b-y)\right]\sin\frac{2n+1}{2a}\pi x.$$

Plug this formula into (14.21c), we get

$$\begin{cases}\sum_{n=0}^{\infty}B_n\cosh\frac{2n+1}{2a}\pi b\sin\frac{2n+1}{2a}\pi x=f(x)\\ \sum_{n=0}^{\infty}A_n\frac{2n+1}{2a}\pi\cosh\frac{2n+1}{2a}\pi b\sin\frac{2n+1}{2a}\pi x=0.\end{cases}$$

So $A_n=0$ and $B_n=\frac{\frac{2}{a}\int_0^a f(x)\sin\frac{2n+1}{2a}\pi x dx}{\cosh\frac{2n+1}{2a}\pi b}$. Thus, choice (2) makes it easier to solve for A_n and B_n. □

14.5.

Proof. Note in equation (14.21), the roles of x and y are symmetric, so the problem can be reduced to solving the following two problems:

$$\begin{cases}\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0, & 0<x<a,0<y<b,\\ u|_{x=0}=0,\ \frac{\partial u}{\partial x}\Big|_{x=a}=0, & 0\le y\le b,\\ u|_{y=0}=\phi(x),\ \frac{\partial u}{\partial y}\Big|_{y=b}=\psi(x), & 0\le x\le a.\end{cases}$$

and

$$\begin{cases}\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0, & 0<x<a,0<y<b,\\ u|_{x=0}=f(y),\ \frac{\partial u}{\partial x}\Big|_{x=a}=g(y), & 0\le y\le b,\\ u|_{y=0}=0,\ \frac{\partial u}{\partial y}\Big|_{y=b}=0, & 0\le x\le a.\end{cases}$$

These two problems can be solved via separation of variables. Then the solution to the original problem is the sum of the respective solutions to the two new problems. □

14.6.

Proof. Suppose $v(x,t)$ is a special solution to the problem

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = f(x,t), & 0 < x < l, t > 0 \\ u|_{x=0} = 0,\ u|_{x=l} = 0, & t \geq 0. \end{cases}$$

Let $w(x,t) = X(x)T(t)$ be a solution to the homogeneous problem

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = 0, & 0 < x < l, t > 0 \\ u|_{x=0} = 0,\ u|_{x=l} = 0, & t \geq 0. \end{cases}$$

Then $w(x,t)$ must have the form of $\sum_{n=1}^{\infty} \left(C_n \sin \frac{n\pi}{l} at + D_n \cos \frac{n\pi}{l} at \right) \sin \frac{n\pi}{l} x$. Let $u(x,t) = v(x,t) + w(x,t)$, then the initial value condition becomes

$$\begin{cases} \sum_{n=1}^{\infty} D_n \sin \frac{n\pi}{l} x = -v(x,0) + \phi(x) \\ \sum_{n=1}^{\infty} C_n \frac{n\pi a}{l} \sin \frac{n\pi}{l} x = -\frac{\partial v(x,t)}{\partial t}\Big|_{t=0} + \psi(x). \end{cases}$$

Using the orthogonality of the eigenfunctions, we have

$$C_n = \frac{2}{n\pi a} \int_0^l \left(-\frac{\partial v(x,t)}{\partial t}\Big|_{t=0} + \psi(x) \right) \sin \frac{n\pi}{l} x dx,\ D_n = \frac{2}{l} \int_0^l [-v(x,0) + \phi(x)] \sin \frac{n\pi}{l} x dx.$$

□

14.7.

Proof. Yes, we can. See §14.6 for detailed discussion. Once we obtain a solution $u_1(x,t)$ to equation (14.81) and a solution $u_2(x,t)$ to equation (14.49), $u(x,t) = u_1(x,t) + u_2(x,t)$ will be a solution to the problem under consideration. □

14.8.

Proof. No, because we want $\{X_n(x)\}$ to be complete. The boundary condition (14.49b) or (14.74) guarantees the self-adjointness of the differential operator associated with the eigenvalue problem satisfied by $X_n(x)$, which implies the completeness of $\{X_n(x)\}$. See Chapter 18, the discussion after Example 18.6 as well as Proposition 4. □

14.9.

Proof. $v(x,t) = -\frac{(l-x)^2}{2l} \mu(t) + \frac{x^2}{2l} \nu(t)$. □

14.10.

Proof. $v(x,t) = \frac{l-x}{l} \mu(t) + \frac{x^2}{2l} \nu(t)$. □

14.11.

Proof. *Step 1.* Find a function $v(x,t)$ satisfying the boundary value condition

$$v(x,t)|_{x=0} = \mu(t),\ v(x,t)|_{x=l} = \nu(t).$$

Step 2. Find a solution $w(x,t)$ to the problem

$$\begin{cases} \frac{\partial^2 w}{\partial t^2} - a^2 \frac{\partial^2 w}{\partial x^2} = f(x,t) - \left[\frac{\partial^2 v}{\partial t^2} - a^2 \frac{\partial^2 v}{\partial x^2} \right], & 0 < x < l, t > 0 \\ w|_{x=0} = 0,\ w|_{x=l} = 0, & t \geq 0 \\ w|_{t=0} = \phi(x) - v(x,0),\ \frac{\partial w}{\partial t}\Big|_{t=0} = \psi(x) - \frac{\partial v(x,t)}{\partial t}\Big|_{t=0}, & 0 \leq x \leq l. \end{cases}$$

This step can be further divided into the following sub-steps.

Step 2.1. Solve the following eigenvalue problem

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = 0,\ X(l) = 0, \end{cases}$$

and obtain a system of orthogonal eigenfunctions $\{X_n(x)\}_{n=1}^\infty = \{\sin \frac{n\pi}{l} x\}_{n=1}^\infty$.

Step 2.2. Expand $w(x,t)$ and $f(x,t) - \left[\frac{\partial^2 v}{\partial t^2} - a^2 \frac{\partial^2 v}{\partial x^2}\right]$ according to eigenfunctions $\{X_n(x)\}$:

$$\begin{cases} w(x,t) = \sum_{n=1}^\infty T_n(t) X_n(x) \\ f(x,t) - \left[\frac{\partial^2 v}{\partial t^2} - a^2 \frac{\partial^2 v}{\partial x^2}\right] = \sum_{n=1}^\infty g_n(t) X_n(x), \end{cases}$$

where $g_n(t) = \frac{2}{l}\int_0^l \left\{ f(x,t) - \left[\frac{\partial^2 v}{\partial t^2} - a^2 \frac{\partial^2 v}{\partial x^2}\right]\right\} \sin \frac{n\pi}{l} x dx$. Plug these formulas back into the partial differential equation for $w(x,t)$:

$$\sum_{n=1}^\infty [T_n''(t) + \lambda_n a^2 T_n(t)] X_n(x) = \sum_{n=1}^\infty g_n(t) X_n(x).$$

Using orthogonality of $\{X_n(x)\}_{n=1}^\infty$, we conclude $T_n''(t) + \lambda_n a^2 T_n(t) = g_n(t)$. Note $w(x,t)$ automatically satisfies the boundary conditions as $X_n(0) = X_n(l) = 0$. To make it further satisfy the initial value condition, we need to expand $\phi(x) - v(x,0)$ and $\psi(x) - \left.\frac{\partial v(x,t)}{\partial t}\right|_{t=0}$ according to eigenfunctions $\{X_n(x)\}$:

$$\begin{cases} \phi(x) - v(x,0) = \sum_{n=1}^\infty a_n X_n(x) \\ \psi(x) - \left.\frac{\partial v(x,t)}{\partial t}\right|_{t=0} = \sum_{n=1}^\infty b_n X_n(x), \end{cases}$$

where $a_n = \frac{2}{l}\int_0^l [\phi(x) - v(x,0)] \sin \frac{n\pi}{l} x dx$ and $b_n = \frac{2}{l}\int_0^l \left[\psi(x) - \left.\frac{\partial v(x,t)}{\partial t}\right|_{t=0}\right] \sin \frac{n\pi}{l} x dx$. Combined, we can have an equation for $T_n(t)$:

$$\begin{cases} T_n''(t) + \lambda_n a^2 T_n(t) = g_n(t) \\ T_n(0) = a_n,\ T_n'(0) = b_n. \end{cases}$$

Once we find $T_n(t)$, the solution to the original PDE can be written as $u(x,t) = v(x,t) + \sum_{n=1}^\infty T_n(t) \sin \frac{n\pi}{l} x$.

Step 2.3. There are many methods to solve the equation

$$\begin{cases} T_n''(t) + \lambda_n a^2 T_n(t) = g_n(t) \\ T_n(0) = a_n,\ T_n'(0) = b_n. \end{cases}$$

We apply method of Green's function to refresh our memory of Chapter 10. The Green's function corresponding to the above initial value problem is

$$\begin{cases} \frac{d^2}{dt^2} g(t;s) + \lambda_n a^2 g(t;s) = \delta(t-s),\ t,s > 0 \\ g(t;s)|_{t<s} = 0,\ \left.\frac{dg(t;s)}{dt}\right|_{t<s} = 0. \end{cases}$$

So $g(x;t)$ has the general form $[A(s)\sin\sqrt{\lambda_n} at + B(s)\cos\sqrt{\lambda_n} at]\eta(t-s)$ where $\eta(\xi) = \begin{cases} 1 & \xi \geq 0 \\ 0 & \xi < 0 \end{cases}$. By the continuity of $g(t;s)$ at $t = s$ and $\left.\frac{dg(t;s)}{dt}\right|_{s-}^{s+} = 1$, we have

$$\begin{cases} A(s)\sin\sqrt{\lambda_n} as + B(s)\cos\sqrt{\lambda_n} as = 0 \\ \sqrt{\lambda_n} a[A(s)\cos\sqrt{\lambda_n} as - B(s)\sin\sqrt{\lambda_n} as] = 1, \end{cases}$$

which implies

$$A(s) = \frac{\cos\sqrt{\lambda_n}as}{\sqrt{\lambda_n}a},\ B(s) = -\frac{\sin\sqrt{\lambda_n}as}{\sqrt{\lambda_n}a}.$$

So

$$\begin{aligned} g(t;s) &= \frac{1}{\sqrt{\lambda_n}a}[\cos\sqrt{\lambda_n}as\sin\sqrt{\lambda_n}at - \sin\sqrt{\lambda_n}as\cos\sqrt{\lambda_n}at]\eta(t-s) \\ &= \frac{1}{\sqrt{\lambda_n}a}\sin\sqrt{\lambda_n}a(t-s)\eta(t-s). \end{aligned}$$

Therefore by formula (10.60) (recall $\lambda_n = \left(\frac{n\pi}{l}\right)^2$)

$$\begin{aligned} T_n(t) &= \int_0^t g(t;s)g_n(s)ds - \left[a_n\frac{dg(t;s)}{ds} - b_n g(t;s)\right]\Big|_{s=0} \\ &= \frac{l}{n\pi a}\int_0^t \sin\frac{n\pi}{l}a(t-s)g_n(s)ds + a_n\cos\frac{n\pi}{l}at + \frac{b_n l}{n\pi a}\sin\frac{n\pi}{l}at. \end{aligned}$$

Step 3. $u(x,t) = v(x,t) + \sum_{n=1}^{\infty} T_n(t)\sin\frac{n\pi}{l}x$ is the solution to the original problem, where

$$T_n(t) = \frac{l}{n\pi a}\int_0^t \sin\frac{n\pi}{l}a(t-s)g_n(s)ds + a_n\cos\frac{n\pi}{l}at + \frac{b_n l}{n\pi a}\sin\frac{n\pi}{l}at$$

with $a_n = \frac{2}{l}\int_0^l[\phi(x) - v(x,0)]\sin\frac{n\pi}{l}xdx$, $b_n = \frac{2}{l}\int_0^l\left[\psi(x) - \frac{\partial v(x,t)}{\partial t}\Big|_{t=0}\right]\sin\frac{n\pi}{l}xdx$, and

$$g_n(t) = \frac{2}{l}\int_0^l\left\{f(x,t) - \left[\frac{\partial^2 v}{\partial t^2} - a^2\frac{\partial^2 v}{\partial x^2}\right]\right\}\sin\frac{n\pi}{l}xdx.$$

□

14.2 Exercise at the end of chapter

1.

Proof. The partial differential equation under consideration is

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = 0, & \\ u|_{x=0} = 0, & \frac{\partial u}{\partial x}\Big|_{x=l} = 0, \\ u|_{t=0} = \frac{F}{ES}x, & \frac{\partial u}{\partial t}\Big|_{t=0} = 0. \end{cases}$$

The corresponding eigenvalue problem is

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = 0,\ X'(l) = 0. \end{cases}$$

The solution is $X_n(x) = \sin\frac{2n+1}{2l}\pi x$ $(n \geq 0)$ with eigenvalue $\lambda_n = \left(\frac{2n+1}{2l}\right)^2$. Expand $\frac{F}{ES}x$ according to $\{X_n(x)\}_{n=1}^{\infty}$, we have the coefficient

$$a_n = \frac{2}{l}\int_0^l \frac{F}{ES}x\sin\frac{2n+1}{2l}\pi xdx = \frac{8Fl}{ES\pi^2}\frac{(-1)^n}{(2n+1)^2}.$$

So $T_n(t)$ satisfies the ODE

$$\begin{cases} T_n''(t) + \lambda_n a^2 T_n(t) = 0 \\ T_n(0) = a_n,\ T_n'(0) = 0. \end{cases}$$

Therefore $T_n(t) = a_n \cos\sqrt{\lambda_n}at$ and

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t)X_n(x) = \frac{8Fl}{ES\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\sin\frac{2n+1}{2l}\pi x\cos\frac{2n+1}{2l}at.$$

□

2.

Proof. The partial differential equation to be solved is

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = 0, & 0 < x < l, t > 0 \\ u|_{x=0} = u_{x=l} = 0, & t \geq 0, \\ u|_{t=0} = \frac{h}{c}x1_{\{0\leq x\leq c\}} + \frac{h}{l-c}(l-x)1_{\{c\leq x\leq l\}}, \ \frac{\partial u}{\partial t}\Big|_{t=0} = 0, & 0 \leq x \leq t. \end{cases}$$

This is a special case of Exercise Problem 14.11, with $f(x,t) = 0$, $\mu(t) = \nu(t) = 0$, $\psi(x) = 0$, and $\phi(x) = \begin{cases} \frac{h}{c}x & 0 \leq x \leq c \\ \frac{h}{l-c}(l-x) & c \leq x \leq l \end{cases}$. Applying the formula we obtained for Exercise Problem 14.11, we have $u(x,t) = \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{l}at \sin\frac{n\pi}{l}x$, where

$$a_n = \frac{2}{l}\int_0^l \phi(x)\sin\frac{n\pi}{l}xdx = \frac{2}{l}\int_0^c \frac{h}{c}x\sin\frac{n\pi}{l}xdx + \frac{2}{l}\int_c^l \frac{h}{l-c}(l-x)\sin\frac{n\pi}{l}xdx = \frac{2hl^2}{c(l-c)(n\pi)^2}\sin\frac{c}{l}n\pi.$$

Therefore $u(x,t) = \frac{2hl^2}{c(l-c)\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{l}c\cos\frac{n\pi}{l}at\sin\frac{n\pi}{l}x$. □

3.

Proof. The partial differential equation under consideration is

$$\begin{cases} \frac{\partial u}{\partial t} - \kappa\frac{\partial^2 u}{\partial x^2} = 0 \\ u|_{x=0} = u|_{x=l} = 0 \\ u|_{t=0} = b\frac{x(l-x)}{l^2}. \end{cases}$$

Plug $T(t)X(x)$ into the differential equation, we get $T'(t)X(t) - T(t)X''(t) = 0$. So the eigenvalue problem associated with this PDE is

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = X(l) = 0. \end{cases}$$

If $\lambda = 0$, $X \equiv 0$. If $\lambda \neq 0$, the solution must have the form $X_n(x) = \sin\sqrt{\lambda_n}x$ with $\lambda_n = \left(\frac{n\pi}{l}\right)^2$ $(n \in \mathbb{N})$. Suppose $u(x,t) = \sum_{n=1}^{\infty} T_n(t)X_n(x)$. Then

$$\begin{cases} \sum_{n=1}^{\infty}[T_n'(t) + \lambda\kappa T_n(t)]X_n(x) = 0 \\ \sum_{n=1}^{\infty} T_n(0)X_n(x) = b\frac{x(l-x)}{l^2}. \end{cases}$$

Using orthogonality of $\{X_n(x)\}_{n=1}^{\infty}$, we conclude

$$\begin{cases} T_n'(t) + \lambda_n\kappa T_n(t) = 0 \\ T_n(0) = a_n, \end{cases}$$

where

$$a_n = \frac{\frac{b}{l^2}\int_0^l x(l-x)X_n(x)dx}{\int_0^l X_n^2(x)dx} = \frac{4b}{n^3\pi^3}[1 + (-1)^{n+1}] = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{8b}{n^3\pi^3} & \text{if } n \text{ is odd.} \end{cases}$$

Therefore,

$$T_n(t) = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{8b}{n^3\pi^3} e^{-\lambda_n \kappa t} & \text{if } n \text{ is odd.} \end{cases}$$

Finally, we can conclude

$$u(x,t) = \frac{8b}{\pi^3}\sum_{n=1}^{\infty} \frac{1}{(2n+1)^3} e^{-\left(\frac{2n+1}{l}\pi\right)^2 \kappa t} \sin\frac{2n+1}{l}\pi x.$$

□

5.

Proof. Plug $u(x,y) = X(x)Y(y)$ into the differential equation, we get $\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)}$. So the associated eigenvalue problem for $Y(y)$ is

$$Y''(y) + \lambda Y(y) = 0$$
$$Y'(0) = Y'(b) = 0.$$

Therefore nontrivial solution is $Y_n(y) = \cos\sqrt{\lambda_n} y$ with $\lambda_n = \left(\frac{n\pi}{b}\right)^2$ $(n \in \mathbb{N})$. Suppose $u(x,y) = \sum_{n=1}^{\infty} X_n(x)Y_n(y)$, we have

$$\begin{cases} \sum_{n=1}^{\infty}[X_n''(x) - \lambda_n X_n(x)]Y_n(y) = 0 \\ \sum_{n=1}^{\infty} X_n(0)Y_n(y) = u_0,\ \sum_{n=1}^{\infty} X_n(a)Y_n(y) = u_0\left[3\left(\frac{y}{b}\right)^2 - 2\left(\frac{y}{b}\right)^3\right]. \end{cases}$$

Using the orthogonality of $\{Y_n(y)\}_{n=1}^{\infty}$, we have

$$\begin{cases} X_n''(x) - \lambda_n X_n(x) = 0 \\ X_n(0) = a_n, \qquad X_n(a) = b_n, \end{cases}$$

where

$$a_n = \frac{\int_0^b u_0 Y_n(y)dy}{\int_0^b Y_n^2(y)dy} = 0,\ b_n = \frac{\int_0^b u_0\left[3\left(\frac{y}{b}\right)^2 - 2\left(\frac{y}{b}\right)^3\right] Y_n(y)dy}{\int_0^b Y_n^2(y)dy} = \begin{cases} 0 & \text{if } n \text{ is even} \\ -\frac{48u_0}{\lambda_n^2 b^4} & \text{if } n \text{ is odd.} \end{cases}$$

Solving the ODE for $X_n(x)$, we get

$$X_n(x) = \begin{cases} 0 & \text{if } n \text{ is even} \\ -\frac{48u_0}{(n\pi)^4 \sinh\frac{n\pi}{b}a} \sinh\frac{n\pi}{b}x & \text{if } n \text{ is odd.} \end{cases}$$

Combined, we conclude

$$u(x,y) = -\frac{48u_0}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{(2n+1)^4}\frac{\sinh\frac{2n+1}{b}\pi x}{\sinh\frac{2n+1}{b}\pi a}\cos\frac{2n+1}{b}\pi y.$$

□

6.

Proof. We apply the formula developed in Exercise Problem 14.11. In this problem's context, we have $\phi(x) = \psi(x) = 0$, $v(x,t) = 0$, and $f(x,t) = bx(l-x)$. So

$$g_n(t) = \frac{2}{l}\int_0^l bx(l-x)\sin\frac{n\pi}{l}xdx = \frac{4bl^2}{(n\pi)^3}[(-1)^{n+1}+1] = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{8bl^2}{(n\pi)^3} & \text{if } n \text{ is odd.} \end{cases}$$

This implies

$$T_n(t) = \frac{l}{n\pi a}\int_0^t \sin\frac{n\pi}{l}a(t-s)ds\cdot\frac{4bl^2}{(n\pi)^3}[(-1)^{n+1}+1] = \frac{4bl^4}{(n\pi)^5a^2}[(-1)^{n+1}+1]\left(1-\cos\frac{n\pi}{l}at\right).$$

Therefore

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t)\sin\frac{n\pi}{l}x = \frac{8bl^4}{\pi^5a^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^5}\sin\frac{2n+1}{l}\pi x\left(1-\cos\frac{2n+1}{l}\pi at\right).$$

Remark 25. *The textbook's answer has a π missing in $\sin\frac{2n+1}{l}\pi x$.*

□

7. (1)

Proof. Let $u(x,y) = X(x)Y(y)$. Then the original equation becomes

$$\begin{cases} X''(x)Y(y) + X(x)Y''(y) = -2 \\ X(0)Y(y) = X(a)Y(y) = 0 \\ X(x)Y(\frac{b}{2}) = X(x)Y(-\frac{b}{2}) = 0. \end{cases}$$

So we can consider the eigenvalue problem

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = X(a) = 0, \end{cases}$$

which has the solution $X_n(x) = \sin\sqrt{\lambda_n}x$ with $\lambda_n = \left(\frac{n\pi}{a}\right)^2$ $(n\in\mathbb{N})$. Suppose $u(x,y) = \sum_{n=1}^{\infty} X_n(x)Y_n(y)$. Then by the orthogonality of $\{X_n(x)\}_{n=1}^{\infty}$, we have

$$\begin{cases} Y_n''(y) - \lambda_n Y_n(y) = \frac{4}{n\pi}[(-1)^n - 1] \\ Y_n(-\frac{b}{2}) = Y_n(\frac{b}{2}) = 0. \end{cases}$$

To find the expression for $Y_n(y)$, we note the Green's function associated with $Y_n(y)$ satisfies

$$\begin{cases} \frac{d^2}{dy^2}g(y;t) - \lambda_n g(y;t) = \delta(y-t) \\ g(-\frac{b}{2};t) = g(\frac{b}{2};t) = 0. \end{cases}$$

Therefore

$$g(y;t) = \begin{cases} A(t)\sinh\sqrt{\lambda_n}(y+\frac{b}{2}), & -\frac{b}{2} < y < t \\ B(t)\sinh\sqrt{\lambda_n}(y-\frac{b}{2}), & t < y < \frac{b}{2}, \end{cases}$$

where the coefficient function $A(t)$ and $B(t)$ are determined by the continuity property of $g(y;t)$ at $y=t$:

$$\begin{cases} A(t)\sinh\sqrt{\lambda_n}(t+\frac{b}{2}) = B(t)\sinh\sqrt{\lambda_n}(t-\frac{b}{2}) \\ \sqrt{\lambda_n}B(t)\cosh\sqrt{\lambda_n}(t-\frac{b}{2}) - \sqrt{\lambda_n}A(t)\cosh\sqrt{\lambda_n}(t+\frac{b}{2}) = 1. \end{cases}$$

Solving this equation gives

$$\begin{cases} A(t) = \frac{1}{\sqrt{\lambda_n}\sinh\sqrt{\lambda_n}b}\sinh\sqrt{\lambda_n}(t-\frac{b}{2}) \\ B(t) = \frac{1}{\sqrt{\lambda_n}\sinh\sqrt{\lambda_n}b}\sinh\sqrt{\lambda_n}(t+\frac{b}{2}). \end{cases}$$

Therefore

$$g(y;t) = \frac{\sinh\sqrt{\lambda_n}(t-\frac{b}{2})\sinh\sqrt{\lambda_n}(y+\frac{b}{2})1_{\{-\frac{b}{2}<y<t\}} + \sinh\sqrt{\lambda_n}(t+\frac{b}{2})\sinh\sqrt{\lambda_n}(y-\frac{b}{2})1_{\{t<y<\frac{b}{2}\}}}{\sqrt{\lambda_n}\sinh\sqrt{\lambda_n}b}$$

and

$$\begin{aligned}
\int_{-\infty}^{\infty} g(y;t)dt &= \int_{-\frac{b}{2}}^{y} \frac{\sinh\sqrt{\lambda_n}(y-\frac{b}{2})}{\sqrt{\lambda_n}\sinh\sqrt{\lambda_n}b}\sinh\sqrt{\lambda_n}(t+\frac{b}{2})dt + \int_{y}^{\frac{b}{2}} \frac{\sinh\sqrt{\lambda_n}(y+\frac{b}{2})}{\sqrt{\lambda_n}\sinh\sqrt{\lambda_n}b}\sinh\sqrt{\lambda_n}(t-\frac{b}{2})dt \\
&= \frac{\sinh\sqrt{\lambda_n}(y-\frac{b}{2})}{\lambda_n\sinh\sqrt{\lambda_n}b}\left[\cosh\sqrt{\lambda_n}(y+\frac{b}{2})-1\right] + \frac{\sinh\sqrt{\lambda_n}(y+\frac{b}{2})}{\lambda_n\sinh\sqrt{\lambda_n}b}\left[1-\cosh\sqrt{\lambda_n}(y-\frac{b}{2})\right] \\
&= \frac{-\sinh\sqrt{\lambda_n}b - \sinh\sqrt{\lambda_n}(y-\frac{b}{2}) + \sinh\sqrt{\lambda_n}(y+\frac{b}{2})}{\lambda_n\sinh\sqrt{\lambda_n}b} \\
&= \frac{-\sinh\sqrt{\lambda_n}b + 2\sinh\frac{\sqrt{\lambda_n}b}{2}\cosh\sqrt{\lambda_n}y}{\lambda_n\sinh\sqrt{\lambda_n}b} \\
&= -\frac{1}{\lambda_n} + \frac{\cosh\sqrt{\lambda_n}y}{\lambda_n\cosh\frac{\sqrt{\lambda_n}b}{2}}.
\end{aligned}$$

Hence

$$\begin{aligned}
Y_n(y) &= \frac{4}{n\pi}[(-1)^n-1]\int_{-\infty}^{\infty} g(y;t)dt = \frac{4}{n\pi}[(-1)^{n+1}+1]\left[\frac{a^2}{(n\pi)^2} - \frac{a^2\cosh\frac{n\pi y}{a}}{(n\pi)^2\cosh\frac{n\pi b}{2a}}\right] \\
&= \begin{cases} \frac{8a^2}{(n\pi)^3}\left[1-\frac{\cosh\frac{n\pi y}{a}}{\cosh\frac{n\pi b}{2a}}\right] & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}
\end{aligned}$$

Combined, we have

$$u(x,t) = \sum_{n=1}^{\infty} X_n(x)Y_n(y) = \frac{8a^2}{\pi^3}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3}\left[1-\frac{\cosh\frac{2n+1}{a}\pi y}{\cosh\frac{2n+1}{2a}\pi b}\right]\sin\frac{2n+1}{a}\pi x.$$

□

(2)

Proof. Similar to part (1), we find the solution to the eigenvalue problem

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = X(a) = 0 \end{cases}$$

as $X_n(x) = \sin\sqrt{\lambda_n}x$ with $\lambda_n = \left(\frac{n\pi}{a}\right)^2$ $(n \in \mathbb{N})$. Expanding $u(x,y)$ according to $\{X_n(x)\}_{n=1}^{\infty}$, we have

$$u(x,y) = \sum_{n=1}^{\infty} Y_n(y)X_n(x),$$

where

$$Y_n(y) = \frac{\int_0^a x^2yX_n(x)dx}{\int_0^a X_n^2(x)dx} = \frac{2}{a}\int_0^a x^2yX_n(x)dx = 2y\left\{\frac{a^2(-1)^{n+1}}{n\pi} + \frac{2a^2}{(n\pi)^3}[(-1)^n-1]\right\}.$$

So the eigenvalue problem associated with $Y_n(x)$ becomes

$$\begin{cases} Y_n''(x) - \lambda_n Y_n(x) = 2y\left\{\frac{a^2(-1)^{n+1}}{n\pi} + \frac{2a^2}{(n\pi)^3}[(-1)^n-1]\right\} \\ Y_n(-\frac{b}{2}) = Y_n(\frac{b}{2}) = 0. \end{cases}$$

The Green's function associated with $Y_n(y)$ is the same as that of part (1):

$$g(y;t) = \frac{\sinh\sqrt{\lambda_n}(t-\frac{b}{2})\sinh\sqrt{\lambda_n}(y+\frac{b}{2})1_{\{-\frac{b}{2}<y<t\}} + \sinh\sqrt{\lambda_n}(t+\frac{b}{2})\sinh\sqrt{\lambda_n}(y-\frac{b}{2})1_{\{t<y<\frac{b}{2}\}}}{\sqrt{\lambda_n}\sinh\sqrt{\lambda_n}b}.$$

Therefore

$$\begin{aligned} Y_n(y) &= \int_{-\infty}^{\infty} 2t \left\{ \frac{a^2(-1)^{n+1}}{n\pi} + \frac{2a^2}{(n\pi)^3}[(-1)^n - 1] \right\} g(y;t)dt \\ &= \frac{2a^2}{n\pi} \left\{ (-1)^{n+1} + \frac{2}{(n\pi)^2}[(-1)^n - 1] \right\} \int_{-\infty}^{\infty} tg(y;t)dt. \end{aligned}$$

Note

$$\begin{aligned} \int_{-\infty}^{\infty} tg(y;t)dt &= \frac{\sinh\sqrt{\lambda_n}(y+\frac{b}{2})}{\sqrt{\lambda_n}\sinh\sqrt{\lambda_n}b} \int_y^{\frac{b}{2}} t\sinh\sqrt{\lambda_n}(t-\frac{b}{2})dt + \frac{\sinh\sqrt{\lambda_n}(y-\frac{b}{2})}{\sqrt{\lambda_n}\sinh\sqrt{\lambda_n}b} \int_{-\frac{b}{2}}^{y} t\sinh\sqrt{\lambda_n}(t+\frac{b}{2})dt \\ &= \frac{\sinh\sqrt{\lambda_n}(y+\frac{b}{2})}{\sqrt{\lambda_n}\sinh\sqrt{\lambda_n}b} \left[\frac{b}{2\sqrt{\lambda_n}} - \frac{y}{\sqrt{\lambda_n}}\cosh\sqrt{\lambda_n}(y-\frac{b}{2}) + \frac{1}{\lambda_n}\sinh\sqrt{\lambda_n}(y-\frac{b}{2}) \right] + \\ &\quad \frac{\sinh\sqrt{\lambda_n}(y-\frac{b}{2})}{\sqrt{\lambda_n}\sinh\sqrt{\lambda_n}b} \left[\frac{b}{2\sqrt{\lambda_n}} + \frac{y\cosh\sqrt{\lambda_n}(y+\frac{b}{2})}{\sqrt{\lambda_n}} - \frac{1}{\lambda_n}\sinh\sqrt{\lambda_n}(y+\frac{b}{2}) \right] \\ &= \frac{b[\sinh\sqrt{\lambda_n}(y+\frac{b}{2}) + \sinh\sqrt{\lambda_n}(y-\frac{b}{2})]}{2\lambda_n\sinh\sqrt{\lambda_n}b} + \frac{y\sinh(-b)\sqrt{\lambda_n}}{\lambda_n\sinh\sqrt{\lambda_n}b} \\ &= \frac{b}{2\lambda_n}\frac{\sinh\sqrt{\lambda_n}y}{\sinh\frac{\sqrt{\lambda_n}b}{2}} - \frac{y}{\lambda_n}. \end{aligned}$$

Therefore

$$\begin{aligned} Y_n(y) &= \frac{2a^2}{n\pi} \left\{ (-1)^{n+1} + \frac{2}{(n\pi)^2}[(-1)^n - 1] \right\} \left(\frac{b}{2\lambda_n}\frac{\sinh\sqrt{\lambda_n}y}{\sinh\frac{\sqrt{\lambda_n}b}{2}} - \frac{y}{\lambda_n} \right) \\ &= \begin{cases} -\frac{2a^2}{n\pi\lambda_n}\left(y - \frac{b}{2}\frac{\sinh\sqrt{\lambda_n}y}{\sinh\frac{\sqrt{\lambda_n}b}{2}} \right) + \frac{4}{(n\pi)^2}\frac{2a^2}{n\pi\lambda_n}\left(y - \frac{b}{2}\frac{\sinh\sqrt{\lambda_n}y}{\sinh\frac{\sqrt{\lambda_n}b}{2}} \right) & \text{if } n \text{ is odd,} \\ \frac{2a^2}{n\pi\lambda_n}\left(y - \frac{b}{2}\frac{\sinh\sqrt{\lambda_n}y}{\sinh\frac{\sqrt{\lambda_n}b}{2}} \right) & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

Hence

$$\begin{aligned} & u(x,y) \\ =& \sum_{n=1}^{\infty} Y_n(y)X_n(x) \\ =& \frac{2a^4}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\left(y - \frac{b}{2}\frac{\sinh\frac{n\pi}{a}y}{\sinh\frac{n\pi}{2a}b} \right)\sin\frac{n\pi}{a}x + \frac{8a^4}{\pi^5}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^5}\left(y - \frac{b}{2}\frac{\sin\frac{2n+1}{a}\pi y}{\sinh\frac{2n+1}{2a}\pi b} \right)\sin\frac{2n+1}{a}\pi x. \end{aligned}$$

Remark 26. *The above solution differs from the textbook's answer by a sign. Check.*

□

9.

Proof. We choose $v(x,t) = \cos\frac{\pi}{l}x\cos\frac{\pi}{l}at$ and suppose $u(x,t) = v(x,t) + w(x,t)$. Then $w(x,t)$ satisfies the equation

$$\begin{cases} \frac{\partial^2 w}{\partial t^2} - a^2\frac{\partial^2 w}{\partial x^2} = 0 \\ w|_{x=0} = 0, & \frac{\partial w}{\partial x}\big|_{x=l} = 0 \\ w(x,0) = 0, & \frac{\partial w}{\partial t}\big|_{t=0} = \sin\frac{\pi}{2l}x. \end{cases}$$

Solving the associated eigenvalue problem

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = 0, X'(l) = 0 \end{cases}$$

gives the eigenfunctions $\{X_n(x)\} = \{\sin\sqrt{\lambda_n}x\}$, with $\lambda_n = \left(\frac{\pi+2n\pi}{2l}\right)^2$. Suppose $w(x,t) = \sum_{n=0}^\infty T_n(t)X_n(x)$, then we must have

$$\begin{cases} \sum_{n=0}^\infty (T_n''(t) + \lambda_n a^2)X_n(x) = 0 \\ \sum_{n=0}^\infty T_n(0)X_n(x) = 0,\ \sum_{n=0}^\infty T_n'(0)X_n(x) = X_0(x). \end{cases}$$

Therefore $T_n(t) \equiv 0$ for $n \geq 1$ and $T_0(t) = \frac{2l}{a\pi}\sin\frac{\pi}{2l}at$. Combined, we conclude $u(x,t) = \cos\frac{\pi}{l}x\cos\frac{\pi}{l}at + \frac{2l}{\pi a}\sin\frac{\pi}{2l}x\sin\frac{\pi}{2l}at$. □

10.

Proof. We choose $v(x,t) = \frac{l-x}{l}Ae^{-\alpha^2\kappa t} + \frac{x}{l}Be^{-\beta^2\kappa t}$ and suppose $u(x,t) = v(x,t) + w(x,t)$. Note $v(0,t) = Ae^{-\alpha^2\kappa t}$ and $v(l,t) = Be^{-\beta^2\kappa t}$, we have the following equation for $w(x,t)$:

$$\begin{cases} \frac{\partial w}{\partial t} - \kappa\frac{\partial^2 w}{\partial x^2} = -\left[\frac{\partial v}{\partial t} - \kappa\frac{\partial^2 v}{\partial x^2}\right] = \frac{l-x}{l}A\alpha^2\kappa e^{-\alpha^2\kappa t} + \frac{x}{l}B\beta^2\kappa e^{-\beta^2\kappa t} = f(x,t) \\ w|_{x=0} = w|_{x=l} = 0 \\ w(x,0) = -v(x,0) = -\frac{l-x}{l}A - \frac{x}{l}B = \phi(x). \end{cases}$$

Solving the eigenvalue problem

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = X(l) = 0 \end{cases}$$

gives the eigenfunctions $\{X_n(x)\}_{n=1}^\infty$, where $X_n(x) = \sin\sqrt{\lambda_n}x$ with $\lambda_n = \left(\frac{n\pi}{l}\right)^2$. Expand $f(x,t)$ according to $\{X_n(x)\}$, we have $f(x,t) = \sum_{n=1}^\infty g_n(t)X_n(x)$ with

$$g_n(t) = \frac{2}{l}\int_0^l \left[\frac{l-x}{l}A\alpha^2\kappa e^{-\alpha^2\kappa t} + \frac{x}{l}B\beta^2\kappa e^{-\beta^2\kappa t}\right]X_n(x)dx.$$

Note $\int_0^l X_n(x)dx = \frac{1}{-\sqrt{\lambda_n}}\cos\sqrt{\lambda_n}x|_{x=0}^l = \frac{(-1)^n-1}{-\frac{n\pi}{l}} = \frac{l[(-1)^{n+1}+1]}{n\pi}$ and

$$\int_0^l xX_n(x)dx = \frac{1}{-\sqrt{\lambda_n}}\int_0^l xd\cos\sqrt{\lambda_n}x = \frac{l(-1)^n}{-\frac{n\pi}{l}} = \frac{l^2}{n\pi}(-1)^{n+1}.$$

Therefore

$$\begin{aligned} g_n(t) &= \frac{2}{l}\int_0^l \left[A\alpha^2\kappa e^{-\alpha^2\kappa t}X_n(x) + \frac{B\beta^2\kappa e^{-\beta^2\kappa t} - A\alpha^2\kappa e^{-\alpha^2\kappa t}}{l}xX_n(x)\right]dx \\ &= \frac{2\kappa}{n\pi}A\alpha^2 e^{-\alpha^2\kappa t} + \frac{2\kappa}{n\pi}B\beta^2 e^{-\beta^2\kappa t}(-1)^{n+1}. \end{aligned}$$

Expand $\phi(x)$ according to $\{X_n(x)\}_{n=1}^\infty$, we have $\phi(x) = \sum_{n=1}^\infty a_nX_n(x)$ where

$$a_n = \frac{2}{l}\int_0^l \left[-A + \frac{A-B}{l}x\right]X_n(x)dx = -\frac{2A}{n\pi} - \frac{2B}{n\pi}(-1)^{n+1}.$$

If we let $w(x,t) = \sum_{n=1}^\infty T_n(t)X_n(x)$, then

$$\begin{cases} \sum_{n=1}^\infty [T_n'(t) + \kappa\lambda_nT_n(t)]X_n(x) = \sum_{n=1}^\infty g_n(t)X_n(x) \\ \sum_{n=1}^\infty T_n(0)X_n(x) = \sum_{n=1}^\infty a_nX_n(x). \end{cases}$$

By orthogonality of $\{X_n(x)\}$, we have

$$\begin{cases} T_n'(t) + \kappa\lambda_nT_n(t) = g_n(t) \\ T_n(0) = a_n, \end{cases}$$

which implies

$$\begin{aligned} T_n(t) &= e^{-\kappa\lambda_n t}\int_0^t g_n(s)e^{\kappa\lambda_n s}ds + a_n \\ &= \frac{2\kappa}{n\pi}\left[A\alpha^2\frac{e^{-\alpha^2\kappa t}-e^{-\kappa\lambda_n t}}{-\alpha^2\kappa+\kappa\lambda_n} + B\beta^2(-1)^{n+1}\frac{e^{-\beta^2\kappa t}-e^{-\kappa\lambda_n t}}{-\beta^2\kappa+\kappa\lambda_n}\right] + a_n \end{aligned}$$

Combined, we can write the solution $u(x,t)$ to the original problem as

$$\left[\frac{l-x}{l}Ae^{-\alpha^2\kappa t}+\frac{x}{l}Be^{-\beta^2\kappa t}\right]+\sum_{n=1}^{\infty}\left\{\frac{2\kappa}{n\pi}\left[A\alpha^2\frac{e^{-\alpha^2\kappa t}-e^{-\kappa\left(\frac{n\pi}{l}\right)^2 t}}{-\alpha^2\kappa+\kappa\left(\frac{n\pi}{l}\right)^2}+B\beta^2(-1)^{n+1}\frac{e^{-\beta^2\kappa t}-e^{-\kappa\left(\frac{n\pi}{l}\right)^2 t}}{-\beta^2\kappa+\kappa\left(\frac{n\pi}{l}\right)^2}\right]+a_n\right\}\sin\frac{n\pi}{l}x.$$

Remark 27. *If we choose* $v(x,t) = A\frac{\sin\alpha(l-x)}{\sin\alpha l}e^{-\alpha^2\kappa t}+B\frac{\sin\beta x}{\sin\beta l}e^{-\beta^2\kappa t}$*, then it's easy to verify* $\frac{\partial v}{\partial t}-\kappa\frac{\partial^2 v}{\partial x^2}=0$*. So this choice of* $v(x,t)$ *simultaneously homogenizes boundary condition and the differential equation, which makes the solution much easier.*

□

15 Orthogonal Curvilinear Coordinates

15.1 Exercise in the text

15.1.

Proof. Suppose λ_m and λ_n are two distinct eigenvalues of the eigenvalue problem (15.41). Let Φ_m be an eigenfunction associated with λ_m and Φ_n an eigenfunction associated with λ_n. Then

$$\Phi_m''(\phi)+\lambda_m\Phi_m(\phi)=0,\ \Phi_n''(\phi)+\lambda_n\Phi_n(\phi)=0.$$

Therefore

$$\begin{aligned} 0 &= \left[\int_0^{2\pi}\Phi_m''(\phi)\Phi_n(\phi)d\phi+\lambda_m\int_0^{2\pi}\Phi_m(\phi)\Phi_n(\phi)d\phi\right]-\left[\int_0^{2\pi}\Phi_n''(\phi)\Phi_m(\phi)d\phi+\lambda_n\int_0^{2\pi}\Phi_n(\phi)\Phi_m(\phi)d\phi\right] \\ &= [\Phi_n(\phi)\Phi_m'(\phi)-\Phi_m(\phi)\Phi_n'(\phi)]|_0^{2\pi}+(\lambda_m-\lambda_n)\int_0^{2\pi}\Phi_m(\phi)\Phi_n(\phi)d\phi \\ &= (\lambda_m-\lambda_n)\int_0^{2\pi}\Phi_m(\phi)\Phi_n(\phi)d\phi. \end{aligned}$$

This implies Φ_m and Φ_n are orthogonal to each other. □

15.2.

Proof.

$$\int_0^{2\pi}\sin m\phi\cos m\phi d\phi=\frac{1}{2}\int_0^{2\pi}\sin(2m\phi)d\phi=\frac{1}{4m}\cos(2m\phi)|_0^{2\pi}=0.$$

□

15.3.

Proof.

$$\int_0^{2\pi}e^{im\phi}(e^{-im\phi})^*d\phi=\int_0^{2\pi}e^{2im\phi}d\phi=\frac{e^{2im\cdot 2\pi}-e^{2im\cdot 0}}{2im}=0.$$

□

15.2 Exercise at the end of chapter

3.

Proof. By formula (15.48), the general solution to the equation is

$$u(r,\phi)=\alpha_0+\beta_0\ln r+\sum_{m=1}^{\infty}(C_{m1}r^m+D_{m1}r^{-m})\sin m\phi+\sum_{m=1}^{\infty}(C_{m2}r^m+D_{m2}r^{-m})\cos m\phi.$$

Suppose the expansion of $f(\phi)$ and $g(\phi)$ for the given eigenfunctions $\{0,\sin m\phi,\cos m\phi\}_{m=1}^{\infty}$ are, respectively,

$$f(\phi)=A_0+\sum_{m=1}^{\infty}(A_m\cos m\phi+B_m\sin m\phi),\ \ g(\phi)=C_0+\sum_{m=1}^{\infty})(C_m\cos m\phi+D_m\sin m\phi).$$

The boundary conditions $u(a,\phi)=f(\phi)$ and $u(b,\phi)=g(\phi)$ gives the equations

$$\begin{cases}\alpha_0+\beta_0\ln a=A_0\\ \alpha_0+\beta_0\ln b=C_0,\end{cases}\quad \begin{cases}C_{m1}a^m+D_{m1}a^{-m}=B_m\\ C_{m1}b^m+D_{m1}b^{-m}=D_m,\end{cases}\quad \begin{cases}C_{m2}a^m+D_{m2}a^{-m}=A_m\\ C_{m2}b^m+D_{m2}b^{-m}=C_m.\end{cases}$$

Solving these equations gives us the expression of $u(r,\phi)$ as

$$\begin{aligned}u(r,\phi)\ \ =\ \ & A_0\frac{\ln b-\ln r}{\ln b-\ln a}+\sum_{m=1}^{\infty}\frac{\left(\frac{r}{b}\right)^m-\left(\frac{b}{r}\right)^m}{\left(\frac{a}{b}\right)^m-\left(\frac{b}{a}\right)^m}(A_m\cos m\phi+B_m\sin m\phi)\\ & -C_0\frac{\ln a-\ln r}{\ln b-\ln a}-\sum_{m=1}^{\infty}\frac{\left(\frac{r}{a}\right)^m-\left(\frac{a}{r}\right)^m}{\left(\frac{a}{b}\right)^m-\left(\frac{b}{a}\right)^m}(C_m\cos m\phi+D_m\sin m\phi).\end{aligned}$$

□

4.

Proof. All the subproblems of this exercise share the same feature. So we first deal with a general problem:

$$\begin{cases}\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f(x,y), & x^2+y^2<a^2,\\ u|_{x^2+y^2=a^2}=0.\end{cases}$$

Similar to the calculations carried out in §15.4, we can use polar coordinate to transform the above problem into the following one:

$$\begin{cases}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial\phi^2}=f(r,\phi), & 0<r<a,\ 0<\phi<2\pi,\\ u(r,\phi)|_{\phi=0}=u(r,\phi)|_{\phi=2\pi}, & 0<r<a,\\ \left.\frac{\partial u(r,\phi)}{\partial\phi}\right|_{\phi=0}=\left.\frac{\partial u(r,\phi)}{\partial\phi}\right|_{\phi=2\pi}, & 0<r<a,\\ u|_{r=a}=0, & 0<\phi<2\pi,\\ u(r,\phi)|_{r=0}\text{ is bounded}, & 0<\phi<2\pi.\end{cases}$$

For the eigenvalue problem

$$\begin{cases}\frac{d^2\Phi}{d\phi^2}+\lambda\Phi=0\\ \Phi(0)=\Phi(2\pi)\\ \Phi'(0)=\Phi'(2\pi),\end{cases}$$

we have eigenfunctions $\{1,\sin m\phi,\cos m\phi\}_{m=1}^{\infty}$ with eigenvalues $\{m^2\}_{m=0}^{\infty}$. Expand $u(r,\phi)$ according to this system of eigenfunctions, we obtain

$$u(r,\phi)=A(r)+\sum_{m=1}^{\infty}B_m(r)\sin m\phi+\sum_{m=1}^{\infty}C_m(r)\cos m\phi.$$

Expand also $f(r,\phi)$ according the above system of eigenfunctions, we have

$$f(r,\phi) = a(r) + \sum_{m=1}^{\infty} b_m(r)\sin m\phi + \sum_{m=1}^{\infty} c_m(r)\cos m\phi.$$

Then the equation $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} = f(r,\phi)$ becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{dA(r)}{r}\right) + \sum_{m=1}^{\infty}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{dB_m(r)}{dr}\right) - \frac{m^2}{r^2}B_m(r)\right]\sin m\phi + \sum_{m=1}^{\infty}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{dC_m(r)}{dr}\right) - \frac{m^2}{r^2}C_m(r)\right]\cos m\phi$$
$$= a(r) + \sum_{m=1}^{\infty} b_m(r)\sin m\phi + \sum_{m=1}^{\infty} c_m(r)\cos m\phi.$$

Therefore, we have the following equations

$$\begin{cases}\frac{1}{r}A'(r) + A''(r) = a(r), \\ A(a) = 0,\end{cases} \quad \text{or equivalently} \quad \begin{cases}[rA'(r)]' = ra(r), \\ A(a) = 0,\end{cases}$$

$$\begin{cases}\frac{1}{r}B'_m(r) + B''_m(r) - \frac{m^2}{r^2}B_m(r) = b_m(r), \quad m \geq 1 \\ B_m(a) = 0,\end{cases}$$

$$\begin{cases}\frac{1}{r}C'_m(r) + C''_m(r) - \frac{m^2}{r^2}C_m(r) = c_m(r), \quad m \geq 1 \\ C_m(a) = 0.\end{cases}$$

(1) $f(r,\phi) = -4$. In this case, $a(r) = -4$ and $b_m(r) = c_m(r) = 0$ ($m \in \mathbb{N}$). It's easy to see $A(r) = -r^2 + a^2$. By Theorem 6.3 and the boundedness of $u(r,\phi)$ at $r = 0$, we conclude $B_m(r)$ and $C_m(r)$ are analytic in $\{x^2 + y^2 < a^2\}$ ($m \in \mathbb{N}$). Since $b_m(r) = c_m(r) = 0$, we can deduce $B_m(r) = C_m(r) = 0$. Combined, we conclude $u(r,\phi) = a^2 - r^2$.

(2) $f(r,\phi) = -4r\sin\phi$. So $a(r) = 0$, $b_1(r) = -4r$, $b_n(r) = 0$ ($n \geq 2$), and $c_m(r) = 0$ ($m \in \mathbb{N}$). Then it's easy to see $A(r) = B_n(r) = C_m(r) = 0$ ($m \in \mathbb{N}$, $n \geq 2$) by an argument similar to that of part (1). To find $B_1(r)$, we consider a general power series $\varphi(r) = \sum_{n=0}^{\infty} \alpha_n r^n$. Then

$$\frac{1}{r}\varphi'(r) + \varphi''(r) - \frac{m^2}{r^2}\varphi(r) = -\frac{\alpha_0 m^2}{r^2} + \frac{\alpha_1(1-m^2)}{r} + \sum_{n=0}^{\infty}\alpha_{n+2}[(n+2)^2 - m^2]r^n.$$

By letting $m = 1$ and the above formula equal to $-4r$, we conclude $\alpha_0 = 0$, α_1 is arbitrary, $\alpha_2 = 0$, $\alpha_3 = -\frac{1}{2}$ and $\alpha_n = 0$ for $n \geq 4$. So $B_1(r) = \alpha_1 r - \frac{1}{2}r^3$. By the boundary condition $B_1(a) = 0$, we have $\alpha_1 = \frac{1}{2}a^2$. So $B_1(r) = \frac{1}{2}(a^2 - r^2)r$ and $u(x,t) = \frac{1}{2}(a^2 - r^2)r\sin\phi$.

(3) Similar to the argument in part (1) and (2), we have $u(r,\phi) = \frac{1}{6}(a^2 - r^2)r^2\sin 2\phi$.

(4) Similar to the argument in part (1) and (2), we have $u(r,\phi) = \frac{1}{2}(a^2 - r^2)r(\sin\phi + \cos\phi)$. □

16 Spherical Functions

16.1 Exercise in the text

16.1.

Proof. Multiply both sides of (16.12a) by $y^*(x)$ and integrate from -1 to 1, we have by integration-by-parts formula

$$0 = \nu(\nu+1)\int_{-1}^{1}|y(x)|^2dx + \int_{-1}^{1}\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right]y^*(x)dx = \nu(\nu+1)\int_{-1}^{1}|y(x)|^2dx - \int_{-1}^{1}(1-x^2)\left|\frac{dy}{dx}\right|^2dx.$$

When $y(x) \not\equiv 0$, $\int_{-1}^{1}(1-x^2)\left|\frac{dy}{dx}\right|^2 dx > 0$ and $\int_{-1}^{1}|y(x)|^2 dx > 0$. So

$$\nu(\nu+1) = \frac{\int_{-1}^{1}(1-x^2)\left|\frac{dy}{dx}\right|^2 dx}{\int_{-1}^{1}|y(x)|^2 dx} > 0.$$

□

16.2.

Proof. We note

$$P_l'(x) = \sum_{n=1}^{l}\frac{1}{(n!)^2}\frac{(l+n)!}{(l-n)!}n\left(\frac{x-1}{2}\right)^{n-1}, \; P_l''(x) = \sum_{n=2}^{l}\frac{1}{(n!)^2}\frac{(l+n)!}{(l-n)!}n(n-1)\left(\frac{x-1}{2}\right)^{n-2}.$$

So $P_l'(1) = \frac{(l+1)!}{(l-1)!} = (l+1)l$ and $P_l''(1) = \frac{1}{(2!)^2}\frac{(l+2)!}{(l-2)!}\cdot 2 = \frac{1}{2}(l+2)(l+1)l(l-1)$. □

16.3.

Proof. By formula (16.19), $P_l'(-x)(-1) = (-1)^l P_l'(x)$ and $P_l''(-x) = (-1)^l P_l''(x)$. So

$$P_l'(-1) = (-1)^{l+1}P_l'(1) = (-1)^{l+1}l(l+1)$$

and

$$P_l''(-1) = (-1)^l P_l''(1) = \frac{(-1)^l}{2}(l+2)(l+1)l(l-1).$$

□

16.4.

Proof. $P_{2l}(x) = \sum_{r=0}^{l}(-1)^r\frac{(4l+2-2r)!}{2^{2l+1}r!(2l-r)!(2l-2r)!}x^{2l-2r}$, so $P_{2l}'(0) = 0$.

$$P_{2l+1}(x) = \sum_{r=0}^{l}(-1)^r\frac{(4l+2-2r)!}{2^{2l+1}r!(2l+1-r)!(2l+1-2r)!}x^{2l+1-2r},$$

so $P_{2l+1}'(0) = (-1)^l\frac{(2l+2)!}{2^{2l+1}(2l)!(l+1)!}$. □

16.5.

Proof. This is straightforward from (16.24). □

16.6.

Proof. By separation of variables, we have the following three eigenvalue problems:

$$\begin{cases}\frac{d}{dr}\left[r^2\frac{dR(r)}{dr}\right] - \lambda R(r) = 0\\ \lim_{r\to\infty}R(r) = 0,\end{cases} \quad \begin{cases}\frac{1}{\sin\theta}\frac{d}{d\theta}\left[\sin\theta\frac{d\Theta(\theta)}{d\theta}\right] + \left[\lambda - \frac{\mu}{\sin^2\theta}\right]\Theta(\theta) = 0\\ \Theta(0), \Theta(\pi) \text{ are bounded,}\end{cases} \quad \begin{cases}\Phi'' + \mu\Phi = 0\\ \Phi(0) = \Phi(2\pi), \; \Phi'(0) = \Phi'(2\pi).\end{cases}$$

For the third eigenvalue problem, by the calculations in §15.4, page 216, the eigenvalues are m^2 ($m \in \mathbb{N}$) with corresponding eigenfunctions $\sin m\phi$ and $\cos m\phi$. For the second eigenvalue problem, by §16.8, the eigenvalues are $\lambda_l = l(l+1)$, $l = m, m+1, m+2, \cdots$ with corresponding eigenfunctions $P_l^m(\cos\theta)$, where P_l^m are the associated Legendre's polynomials. Finally, for each given λ_l, the calculation on page 236 shows the first eigenvalue problem has eigenfunction r^{-l-1}. Combined, we conclude the general solution has the form of

$$u(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=0}^{l} r^{-l-1}P_l^m(\cos\theta)[A_{lm}\cos m\phi + B_{lm}\sin m\phi].$$

To determine A_{lm} and B_{lm}, we expand $f(\theta,\phi)$ according to $\{P_l^m(\cos\theta)e^{im\phi}\}$, then use the fact $u(a,\theta,\phi) = f(\theta,\phi)$ and the orthogonality of $\{P_l^m(\cos\theta)e^{im\phi}\}$. □

16.7.

Proof. Similar to the calculations in §16.7 and Exercise Problem 16.6, we can conclude the general solution has the form of

$$u(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=0}^{l} P_l^m(\cos\theta)\{r^l[A_{lm}\cos m\phi + B_{lm}\sin m\phi] + r^{-l-1}[C_{lm}\cos m\phi + D_{lm}\sin m\phi]\}.$$

Then we expand $f(\theta,\phi)$ and $g(\theta,\phi)$ according to $\{P_l^m(\cos\theta)e^{im\phi}\}$ to determine A_{lm}, B_{lm}, C_{lm}, and D_{lm}. □

16.2 Exercise at the end of chapter

17 Cylinder Functions

17.1 Exercise in the text

17.1.

Proof.

$$\begin{aligned}\cos(\nu\pi)N_\nu(z) + \sin(\nu\pi)J_\nu(z) &= \frac{\cos^2(\nu\pi)J_\nu(z) - \cos(\nu\pi)J_{-\nu}(z) + \sin^2(\nu\pi)J_\nu(z)}{\sin(\nu\pi)} \\ &= \frac{J_\nu(z) - \cos(\nu\pi)J_{-\nu}(z)}{\sin(\nu\pi)} = N_{-\nu}(z).\end{aligned}$$

By noting $J_{\pm\nu}(ze^{m\pi i}) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k\pm\nu+1)}\left(\frac{z}{2}\right)^{2k\pm\nu}e^{\pm\nu m\pi i} = J_{\pm}(z)e^{\pm\nu m\pi i}$, we have

$$\begin{aligned}N_\nu(ze^{m\pi i}) &= \frac{\cos(\nu\pi)J_\nu(ze^{m\pi i}) - J_{-\nu}(ze^{m\pi i})}{\sin(\nu\pi)} \\ &= \frac{\cos(\nu\pi)J_\nu(z)e^{m\pi i} - \cos(\nu\pi)J_\nu(z)e^{-\nu m\pi i} + \cos(\nu\pi)J_\nu(z)e^{-\nu m\pi i} - J_{-\nu}(z)e^{-m\pi i}}{\sin(\nu\pi)} \\ &= 2i\sin(m\nu\pi)\cot(\nu\pi)J_v(z) + e^{-m\nu\pi i}N_\nu(z).\end{aligned}$$

The equality $N_{-\nu}(ze^{m\pi i}) = e^{-m\nu\pi i}N_{-\nu}(z) + 2i\sin(m\nu\pi)\csc(\nu\pi)J_\nu(z)$ can be proved similarly. □

17.2.

Proof. If $N_\nu(x)$ and $J_\nu(x)$ have a common zero, then their Wronskian $\begin{vmatrix}J_\nu(x) & N_\nu(x)\\ J'_\nu(x) & N'_\nu(x)\end{vmatrix}$ will vanish at that zero, which contradicts with the fact that J_ν and N_ν are linearly independent (see §6.4, page 79). □

17.3.

Proof. We note

$$\begin{aligned}\frac{d}{dx}[x^\nu N_\nu(x)] &= \frac{d}{dx}\left[\frac{\cos(\nu\pi)x^\nu J_\nu(x) - x^\nu J_{-\nu}(x)}{\sin(\nu\pi)}\right] = \frac{\cos(\nu\pi)x^\nu J_{\nu-1}(x) - \frac{d}{dx}[x^{-(-\nu)}J_{(-\nu)}(x)]}{\sin(\nu\pi)} \\ &= \frac{\cos(\nu\pi)x^\nu J_{\nu-1}(x) + x^\nu J_{-\nu+1}(x)}{\sin(\nu\pi)} = x^\nu\frac{\cos(\nu-1)\pi J_{\nu-1}(x) - J_{-(\nu-1)}(x)}{\sin(\nu-1)\pi} \\ &= x^\nu N_{\nu-1}(x),\end{aligned}$$

and

$$\begin{aligned}\frac{d}{dx}[x^{-\nu}N_\nu(x)] &= \frac{d}{dx}\left[\frac{\cos(\nu\pi)x^{-\nu}J_\nu(x) - x^{-\nu}J_{-\nu}(x)}{\sin(\nu\pi)}\right] = \frac{\cos(\nu\pi)(-x^{-\nu})J_{\nu+1}(x) - x^{-\nu}J_{-\nu-1}(x)}{\sin(\nu\pi)} \\ &= \frac{-\cos(\nu+1)\pi x^{-\nu}J_{\nu+1}(x) + x^{-\nu}J_{-\nu-1}(x)}{\sin(\nu+1)\pi} = -x^{-\nu}N_{\nu+1}(x).\end{aligned}$$

□

17.2 Exercise at the end of chapter

18 Summary of Separation of Variables

18.1 Exercise in the text

18.1.

Proof. For $\alpha \in \mathbb{C}$, we have

$$0 \le \|f - \alpha g\|^2 = \|f\|^2 - 2\mathrm{Re}(f, \alpha g) + |\alpha|^2 \|g\|^2.$$

If $g \neq 0$, we pick $\alpha = \frac{(f,g)}{\|g\|^2}$, from which the Schwarz inequality is immediate. □

18.2.

Proof. Divide both sides of the equation by $a(x)$, we get $y'' + \frac{b(x)}{a(x)} y' + \frac{c(x) - \lambda d(x)}{a(x)} y = 0$. Multiply both sides by $e^{\int \frac{b(x)}{a(x)} dx}$, we get

$$\frac{d}{dx}\left[e^{\int \frac{b(x)}{a(x)} dx} \frac{dy}{dx}\right] + \left\{\lambda \left[-\frac{d(x)}{a(x)} e^{\int \frac{b(x)}{a(x)} dx}\right] - \left[-\frac{c(x)}{a(x)} e^{\int \frac{b(x)}{a(x)} dx}\right]\right\} y = 0.$$

Therefore, we can set

$$p(x) = e^{\int \frac{b(x)}{a(x)} dx}, \ \rho(x) = -\frac{d(x)}{a(x)} e^{\int \frac{b(x)}{a(x)} dx} = -\frac{d(x)}{a(x)} p(x), \ q(x) = -\frac{c(x)}{a(x)} e^{\int \frac{b(x)}{a(x)} dx} = -\frac{c(x)}{a(x)} p(x).$$

□

18.2 Exercise at the end of chapter

1. We apply the result obtained in Exercise Problem 18.2.

(1)

Proof. Multiplying both sides of $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + (x + \lambda)y = 0$ by x, we get $\frac{d}{dx}\left[x^2 \frac{dy}{dx}\right] + (\lambda x + x^2) y = 0$. □

(2)

Proof. We have $p(x) = e^{\int \frac{a - bx}{x(1-x)} dx} = x^a (1-x)^{b-a}$. So $q(x) = 0$ and $\rho(x) = -x^{a-1}(1-x)^{b-a-1}$. □

(3)

Proof. Multiplying both sides of the equation by e^{-x}, we get $\frac{d}{dx}\left[x e^{-x} \frac{dy}{dx}\right] + \lambda e^{-x} y = 0$. □

(4)

Proof. Multiplying both sides of the equation by e^{-x^2}, we get $\frac{d}{dx}\left[e^{-x^2} \frac{dy}{dx}\right] + 2\lambda e^{-x^2} y = 0$. □

2.

Proof. The key idea is to make a change of variable $x = x(r)$ so that $r\frac{dR}{dr} = \frac{dR}{dx}$. This implies $dx = \frac{dr}{r}$, so $x(r) = \ln r$. Plugging $r = e^x$ into the original equation, we have

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{\lambda}{r^2}R \quad = \quad \frac{1}{e^x}\frac{d}{e^x dx}\frac{dR}{dx} + \lambda e^{-2x}R = 0,$$

which is simply the equation $\frac{d^2R(x)}{dx^2} + \lambda R(x) = 0$. This reminds us of the eigenvalue problem (14.3) (§14.1, page 186), only that a is not zero. So we make a further change of variable: $x = z + \ln a$, then we have the following eigenvalue problem

$$\begin{cases} \frac{d^2R(z)}{dz^2} + \lambda R(z) = 0 \\ R(0) = 0, \ R(\ln b - \ln a) = 0. \end{cases}$$

The above new eigenvalue problem is shown in §14.1 to have the solution $\lambda_n = \left(\frac{n\pi}{\ln b - \ln a}\right)^2$, $R_n(z) = \sin\left(\frac{n\pi}{\ln b - \ln a}z\right)$, $n = 1, 2, 3, \cdots$. Changing back to the original variable r, we have $R_n(r) = \sin\left(\frac{\ln r - \ln a}{\ln b - \ln a}n\pi\right)$. □

3.

Proof. Suppose we have two distinct eigenvalues λ_1 and λ_2, with their corresponding eigenfunctions $y_1(x)$ and $y_2(x)$, respectively. Then from the equation $\frac{d}{dx}\left[p(x)\frac{dy_1(x)}{dx}\right] + [\lambda_1\rho(x) - q(x)]y_1(x) = 0$, we have

$$\begin{aligned} 0 &= \int_a^b \left\{ y_2(x)\frac{d}{dx}\left[p(x)\frac{dy_1(x)}{dx}\right] + [\lambda_1\rho(x) - q(x)]y_1(x)y_2(x) \right\} dx \\ &= y_2(x)p(x)y_1'(x)|_a^b + \int_a^b [\lambda_1\rho(x) - q(x)]y_1(x)y_2(x)dx. \end{aligned}$$

By symmetry, we have

$$y_1(x)p(x)y_2'(x)|_a^b + \int_a^b [\lambda_2\rho(x) - q(x)]y_1(x)y_2(x)dx = 0.$$

Taking the difference of these two equations and using the condition $p(a) = p(b)$, we have $(p_0 := p(a) = p(b))$

$$\begin{aligned} 0 &= p_0[y_2(x)y_1'(x) - y_2'(x)y_1(x)]|_a^b + \int_a^b (\lambda_1 - \lambda_2)\rho(x)y_1(x)y_2(x)dx \\ &= p_0\left(\begin{vmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{vmatrix} - 1\right)y_1'(a)y_2(a) - p_0\left(\begin{vmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{vmatrix} - 1\right)y_1(a)y_2'(a) + (\lambda_1 - \lambda_2)\int_a^b \rho(x)y_1(x)y_2(x)dx. \end{aligned}$$

So if $\begin{vmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{vmatrix} = 1$, we must have $\int_a^b \rho(x)y_1(x)y_2(x)dx = 0$ since $\lambda_1 - \lambda_2 \neq 0$. □

4.

Proof. Suppose $u = \sum_k \alpha_k\Phi_k$, then

$$\nabla^2 u = \sum_k \alpha_k\nabla^2\Phi_k = \sum_k \alpha_k(-\lambda_k)\Phi_k = -f = -\sum_k A_k\Phi_k.$$

Comparing the coefficients, we conclude $\alpha_k = \frac{A_k}{\lambda_k}$. So $u = \sum_k \frac{A_k}{\lambda_k}\Phi_k$. □

5.

Proof. We first solve the eigenvalue problem

$$\begin{cases}\nabla^2\Phi(x,y)+\lambda\Phi(x,y)=0\\ \Phi(0,y)=\Phi(a,y)=\Phi(x,0)=\Phi(x,b)=0.\end{cases}$$

Suppose $\Phi(x,y)=X(x)Y(y)$, then we have $\frac{X''(x)}{X(x)}+\frac{Y''(y)}{Y(y)}+\lambda=0$. Solving two separate eigenvalue problems

$$\begin{cases}X''(x)+\alpha X(x)=0\\ X(0)=X(a)=0,\end{cases}\qquad \begin{cases}Y''(y)+\beta Y(y)=0\\ Y(0)=Y(b)=0,\end{cases}$$

we have $\alpha_m=\left(\frac{m\pi}{a}\right)^2$ with $X_m(x)=\sin\frac{m\pi x}{a}$ and $\beta_n=\left(\frac{n\pi}{b}\right)^2$ with $Y_n(y)=\sin\frac{n\pi y}{b}$ $(m,n\in\mathbb{N})$. So the eigenvalue $\lambda_{mn}=\alpha_m+\beta_n=\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2$ and by the result of Problem 4, we have

$$u(x,y)=\sum_{m,n=1}^{\infty}\frac{A_{mn}}{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b},$$

where $A_{mn}=\frac{4}{ab}\int_0^a\sin\frac{m\pi x}{a}dx\int_0^b f(x,y)\sin\frac{n\pi y}{b}dy$. □

19 Applications of Integral Transforms

1.

Proof. Let $U(x,p)=\int_0^\infty u(x,t)e^{-pt}dt$. Then

$$\begin{aligned}0&=\int_0^\infty\frac{\partial u(x,t)}{\partial t}e^{-pt}dt-\kappa\int_0^\infty\frac{\partial^2u(x,t)}{\partial x^2}e^{-pt}dt\\ &=u(x,t)e^{-pt}|_0^\infty+p\int_0^\infty e^{-pt}u(x,t)dt-\kappa\frac{\partial^2U(x,p)}{\partial x^2}\\ &=pU(x,p)-\kappa\frac{\partial^2U(x,p)}{\partial x^2}.\end{aligned}$$

So the general solution for $U(x,p)$ is $C_1(p)e^{\sqrt{\frac{p}{\kappa}}x}+C_2(p)e^{-\sqrt{\frac{p}{\kappa}}x}$. By the boundedness of $u|_{x\to\infty}$, we have the boundedness of $U(x,p)|_{x\to\infty}$. So we must have $C_1(p)\equiv 0$. Since $U(0,p)=u_0\int_0^\infty e^{-pt}dt=\frac{u_0}{p}$, we have $U(x,p)=\frac{u_0}{p}e^{-\sqrt{\frac{p}{\kappa}}x}$. By Example 9.8, $u(x,t)=u_0\text{erfc}\frac{x}{2\sqrt{\kappa t}}$, which satisfies $u|_{t=0}=0$. □

2.

Proof. The partial differential equation for the problem is

$$\begin{aligned}&\frac{\partial u}{\partial t}-\kappa\frac{\partial^2u}{\partial x^2}=0,\ x>0,\ t>0\\ &u|_{t=0}=0,\ x<0\\ &u|_{t=0}=u_0,\ x>0.\end{aligned}$$

Let $U(x,p)=\int_0^\infty u(x,t)e^{-pt}dt$. Then (intuitively, $u(x,t)$ should approach to $\frac{u_0}{2}$ as $t\to\infty$ and is hence bounded)

$$\begin{aligned}0&=\int_0^\infty\frac{\partial u(x,t)}{\partial t}e^{-pt}dt-\kappa\int_0^\infty\frac{\partial^2u(x,t)}{\partial x^2}e^{-pt}dt\\ &=u(x,t)e^{-pt}|_0^\infty+p\int_0^\infty e^{-pt}u(x,t)dt-\kappa\frac{\partial^2U(x,p)}{\partial x^2}\\ &=-u(x,0)+pU(x,p)-\kappa\frac{\partial^2U(x,p)}{\partial x^2}.\end{aligned}$$

So we have the ODE for $U(x,p)$: $\frac{\partial^2 U(x,p)}{\partial x^2} - \frac{p}{\kappa}U(x,p) = -\frac{u_0 1_{\{x>0\}}}{\kappa}$. Using the method of Green's function, we can obtain (see Example 10.8)

$$U(x,p) = \frac{1}{2\sqrt{\frac{p}{\kappa}}}\int_{-\infty}^{\infty} e^{-\sqrt{\frac{p}{\kappa}}|x-t|}\frac{u_0 1_{\{t>0\}}}{\kappa}dt = \frac{u_0}{2\sqrt{p\kappa}}\int_0^{\infty} e^{-\sqrt{\frac{p}{\kappa}}|x-t|}dt = \begin{cases} \frac{u_0}{2p}\exp\{\sqrt{\frac{p}{\kappa}}x\} & x<0, \\ \frac{u_0}{p} - \frac{u_0}{2p}\exp\{-\sqrt{\frac{p}{\kappa}}x\} & x>0. \end{cases}$$

By Example 9.8, we can obtain the formula for $u(x,t)$ as

$$u(x,t) = \begin{cases} \frac{u_0}{2}\mathrm{erfc}\left(-\frac{x}{2\sqrt{\kappa t}}\right) & x<0, \\ u_0 - \frac{u_0}{2}\mathrm{erfc}\left(\frac{x}{2\sqrt{\kappa t}}\right) & x>0. \end{cases}$$

□

3.

Proof. Let $U(x,p) = \int_0^\infty u(x,t)e^{-pt}dt$. Then the equation $\frac{\partial u}{\partial t} - \kappa\frac{\partial^2 u}{\partial x^2} = 0$ gives us $\frac{\partial^2 U(x,p)}{\partial x^2} - \frac{p}{\kappa}U(x,p) = 0$, whose general solution is $C_1(p)e^{\sqrt{\frac{p}{\kappa}}x} + C_2(p)e^{-\sqrt{\frac{p}{\kappa}}x}$. For convenience of applying the boundary conditions, we can write $U(x,p)$ as $C_1(p)\sinh\sqrt{\frac{p}{\kappa}}x + C_2(p)\sinh\sqrt{\frac{p}{\kappa}}(l-x)$. Then

$$U(0,p) = C_2(p)\sinh\sqrt{\frac{p}{\kappa}}l = \int_0^\infty Ae^{-\kappa\alpha^2 t}e^{-pt}dt = \frac{A}{p+\kappa\alpha^2}$$

and

$$U(l,p) = C_1(p)\sinh\sqrt{\frac{p}{\kappa}}l = \int_0^\infty Be^{-\kappa\beta^2 t}e^{-pt}dt = \frac{B}{p+\kappa\beta^2}.$$

Therefore, $U(x,p) = \frac{A}{p+\alpha^2\kappa}\frac{\sinh\sqrt{p/\kappa}(l-x)}{\sinh\sqrt{p/\kappa}l} + \frac{B}{p+\beta^2\kappa}\frac{\sinh\sqrt{p/\kappa}x}{\sinh\sqrt{p/\kappa}l}$. □

4.

Proof. Define $U(k,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty u(x,t)e^{-ikx}dx$ and $F(k,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty f(x,t)e^{-ikx}dx$. Then we have

$$\begin{cases} \frac{\partial^2 U(k,t)}{\partial t^2} + a^2k^2U(k,t) = F(k,t) \\ U(k,0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \phi(x)e^{-ikx}dx := \Phi(k) \\ \left.\frac{\partial U(k,t)}{\partial t}\right|_{t=0} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \psi(x)e^{-ikx}dx := \Psi(k). \end{cases}$$

The solution of the above problem can be obtained by superposition of solutions to the following two problems:

$$\begin{cases} \frac{\partial^2 U(k,t)}{\partial t^2} + a^2k^2U(k,t) = 0 \\ U(k,0) = \Phi(k), \ \left.\frac{\partial U(k,t)}{\partial t}\right|_{t=0} = \Psi(k), \end{cases}$$

and

$$\begin{cases} \frac{\partial^2 U(k,t)}{\partial t^2} + a^2k^2U(k,t) = F(k,t) \\ U(k,0) = 0, \ \left.\frac{\partial U(k,t)}{\partial t}\right|_{t=0} = 0. \end{cases}$$

To solve the first problem, we note the general solution to the homogeneous differential equation is $U(k,t) = C_1(k)\sin(akt) + C_2(k)\cos(akt)$. The initial conditions dictate $C_2(k) = \Phi(k)$ and $C_1(k) = \frac{\Psi(k)}{ak}$. So the solution to the first problem is

$$U(k,t) = \frac{\Psi(k)}{ak}\sin(akt) + \Phi(k)\cos(akt).$$

To solve the second problem, we apply the method of Green's function and obtain (see formula (10.34))

$$U(k,t) = \frac{1}{ak}\int_0^t F(k,\xi)\sin(ak(t-\xi))d\xi.$$

Combined, we conclude the solution to the original non-homogenous second order ODE with non-homogeneous initial conditions has the form of

$$U(k,t) = \Phi(k)\cos(akt) + \frac{\Psi(k)}{ak}\sin(akt) + \frac{1}{ak}\int_0^t F(k,\xi)\sin(ak(t-\xi))d\xi.$$

To find the inverse Fourier transform of $\Phi(k)\cos(akt)$, we note

$$\begin{aligned}\Phi(k)\cos(akt) &= \frac{e^{iakt}+e^{-iakt}}{2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(x)e^{-ikx}dx \\ &= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(x)e^{-ik(x-at)}dx + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(x)e^{-ik(x+at)}dx\right] \\ &= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(y+at)e^{-iky}dx + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(y-at)e^{-iky}dx\right].\end{aligned}$$

So $\mathcal{F}^{-1}[\Phi(k)\cos(akt)] = \frac{1}{2}[\phi(x+at)+\phi(x-at)]$.

To find the inverse Fourier transform of $\frac{\Psi(k)}{ak}\sin(akt)$, we assume $\mathcal{F}^{-1}\left[\frac{\Psi(k)}{ak}\sin(akt)\right] = h(x,t)$. Then

$$\Psi(k)\cos(akt) = \frac{d}{dt}\left[\frac{\Psi(k)}{ak}\sin(akt)\right] = \frac{d}{dt}\mathcal{F}(h(x,t)) = \mathcal{F}\left(\frac{\partial h(x,t)}{\partial t}\right).$$

Using the result for $\Phi(k)\cos(akt)$, we have

$$\frac{\partial h(x,t)}{\partial t} = \frac{\psi(x+at)+\psi(x-at)}{2}.$$

So there exists some function $l(x)$ such that

$$h(x,t) = \frac{\int_{x-at}^{x+at}\psi(\xi)d\xi}{2a} + l(x).$$

Once we "guessed" out the form of $h(x,t)$, we can verify easily that $\mathcal{F}\left[\frac{\int_{x-at}^{x+at}\psi(\xi)d\xi}{2a}\right] = \frac{\Psi(k)}{ak}\sin(akt)$, so $l(x)\equiv 0$ and $\mathcal{F}^{-1}\left(\frac{\Psi(k)}{ak}\sin(akt)\right) = \frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)d\xi$.

To find the inverse Fourier transform of $\frac{1}{ak}\int_0^t F(k,\xi)\sin(ak(t-\xi))d\xi$, we suppose

$$\mathcal{F}^{-1}\left[\frac{1}{ak}\int_0^t F(k,\xi)\sin(ak(t-\xi))d\xi\right] = H(x,t).$$

Then

$$\begin{aligned}\mathcal{F}\left(\frac{\partial H(x,t)}{\partial t}\right) &= \frac{\partial}{\partial t}\mathcal{F}(H(x,t)) = \int_0^t F(k,\tau)\cos(ak(t-\tau))d\tau \\ &= \int_0^t F(k,\tau)\frac{e^{iak(t-\tau)}+e^{-iak(t-\tau)}}{2}d\tau.\end{aligned}$$

By the convolution theorem of Fourier transform, we have

$$\begin{aligned}F(k,\tau)e^{iak(t-\tau)} &= \mathcal{F}(f(x,\tau))\mathcal{F}(\delta(x+a(t-\tau))) = \mathcal{F}\left(\int_{-\infty}^{\infty}f(x-\xi,\tau)\delta(\xi+a(t-\tau))d\xi\right) \\ &= \mathcal{F}(f(x+a(t-\tau),\tau)),\end{aligned}$$

$$\begin{aligned} F(k,\tau)e^{-iak(t-\tau)} &= \mathcal{F}(f(x,\tau))\mathcal{F}(\delta(x-a(t-\tau))) = \mathcal{F}\left(\int_{-\infty}^{\infty} f(x-\xi,\tau)\delta(\xi-a(t-\tau))d\xi\right) \\ &= \mathcal{F}(f(x-a(t-\tau),\tau)). \end{aligned}$$

So we have

$$\mathcal{F}\left(\frac{\partial H(x,t)}{\partial t}\right) = \int_0^t \frac{\mathcal{F}(f(x+a(t-\tau),\tau)) + \mathcal{F}(f(x-a(t-\tau),\tau))}{2} d\tau,$$

which implies

$$\frac{\partial H(x,t)}{\partial t} = \int_0^t \frac{f(x+a(t-\tau),\tau) + f(x-a(t-\tau),\tau)}{2} d\tau.$$

Therefore, there exists some function $l(x)$ such that

$$H(x,t) = l(x) + \frac{1}{2a}\int_0^t \int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi,\tau)d\xi d\tau.$$

Now let's wave hand and assume $l(x) \equiv 0$, we then have the solution to the original problem

$$u(x,t) = \frac{1}{2}[\phi(x+at) + \phi(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at} \psi(\xi)d\xi + \frac{1}{2a}\int_0^t \int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi,\tau)d\xi d\tau.$$

Remark 28. *It'll be nice if we can find an easy way to show $l(x) \equiv 0$ in the above calculation. We leave this to the next version of the solution manual.*

□

5.

Proof. We define

$$\begin{aligned} U(k,m,t) &= \frac{1}{2\pi}\int_{\mathbb{R}^2} u(x,y,t)e^{-ikx-imy}dxdy, \\ \Phi(k,m) &= \frac{1}{2\pi}\int_{\mathbb{R}^2} \phi(x,y)e^{-ikx-imy}dxdy, \end{aligned}$$

and

$$\Psi(k,m) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \psi(x,y)e^{-ikx-imy}dxdy.$$

Then the original problem is converted after Fourier transform into the following problem

$$\begin{cases} \frac{\partial^2 U(k,m,t)}{\partial t^2} + a^2[k^2+m^2]U(k,m,t) = 0 \\ U(k,m,0) = \Phi(x,y), \ \ \frac{\partial U}{\partial t}\big|_{t=0} = \Psi(k,m). \end{cases}$$

Then it's easy to deduce that

$$U(k,m,t) = \Phi(x,y)\cos(\sqrt{k^2+m^2}at) + \frac{\Psi(x,y)}{a\sqrt{k^2+m^2}}\sin(\sqrt{k^2+m^2}at)$$

Suppose $\mathcal{F}^{-1}[\frac{\sin(\sqrt{k^2+m^2}at)}{a\sqrt{k^2+m^2}}] = h(x,y,t)$. Then we have by convolution theorem of Fourier transform

$$\begin{aligned} U(k,m,t) &= \mathcal{F}[\phi(x,y)]\frac{\partial}{\partial t}\mathcal{F}[h(x,y,t)] + \mathcal{F}[\psi(x,y)]\mathcal{F}[h(x,y,t)] \\ &= \mathcal{F}\left[\phi * \frac{\partial}{\partial t}h(\cdot,\cdot,t) + \psi * h(\cdot,\cdot,t)\right]. \end{aligned}$$

Hence, $u(x,y,t)$ can be written as

$$u(x,y,t) = \int_{\mathbb{R}^2} \psi(x',y')h(x-x',y-y',t)dx'dy' + \frac{\partial}{\partial t}\int_{\mathbb{R}^2} \phi(x',y')h(x-x',y-y',t)dx'dy'.$$

To find $h(x,y,t)$, ... (to be continued) □

20 Method of Green's Function

1.

Proof. □

21 Introduction to Calculus of Variation

21.1 Exercise in the text

21.1.

Proof. We follow the line of reasoning in Gelfand and Fomin [4], §35. Assume the integration region R stays fixed while the function $u(x_1, \cdots, x_n)$ goes into

$$u^*(x_1, \cdots, x_n) = u(x_1, \cdots, x_n) + \varepsilon\phi(x_1, \cdots, x_n) + \cdots,$$

where the dots denote terms of order higher than 1 relative to ε. By the variation δJ of the function $J[u] = \int \cdots \int F(x_1, \cdots, x_n, u, u_{x_1}, \cdots, u_{x_n}) dx_1 \cdots dx_n$, corresponding to the transformation $u \to u^*$, we mean the principle linear part (in ε) of the differences $J[u^*] - J[u]$. For simplicity, we write $u(x)$, $\phi(x)$ instead of $u(x_1, \cdots, x_n)$, $\phi(x_1, \cdots, x_n)$, dx instead of $dx_1 \cdots dx_n$, etc. Then, using Taylor's theorem, we find that

$$\begin{aligned} & J[u^*] - J[u] \\ = & \int_R \{F[x, u(x) + \varepsilon\phi(x), u_{x_1}(x) + \varepsilon\phi_{x_1}(x), \cdots, u_{x_n}(x) + \varepsilon\phi_{x_n}(x)] - F[x, u(x), u_{x_1}(x), \cdots, u_{x_n}(x)]\} dx \\ = & \varepsilon \int_R \left(F_u\phi + \sum_{i=1}^n F_{u_{x_i}}\phi_{x_i} \right) dx + \cdots, \end{aligned}$$

where the dots again denote terms of order higher than 1 relative to ε. It follows that

$$\delta J = \varepsilon \int_R \left(F_u\phi + \sum_{i=1}^n F_{u_{x_i}}\phi_{x_i} \right) dx$$

is the variation of the functional $J[u]$.

Next, we try to represent the variation of $J[u]$ as an integral of an expression of the form

$$G(x)\phi(x) + \mathrm{div}(\cdots),$$

i.e., we try to transform δJ in such a way that the derivatives ϕ_{x_i} only appear in a combination of terms which can be written as divergence. To achieve this, we replace $F_{u_{x_i}}\phi_{x_i}(x)$ by $\frac{\partial}{\partial x_i}[F_{u_{x_i}}\phi(x)] - \frac{\partial F_{u_{x_i}}}{\partial x_i}\phi(x)$ and obtain

$$\delta J = \varepsilon \int_R \left(F_u - \sum_{i=1}^n \frac{\partial}{\partial x_i} F_{u_{x_i}} \right) \phi(x) dx + \varepsilon \int_R \sum_{i=1}^n \frac{\partial}{\partial x_i}[F_{u_{x_i}}\phi(x)] dx.$$

This expression for δJ has the important feature that its second term is the integral of a divergence, and hence can be reduced to an integral over the boundary Γ of the integration region. In fact, let $d\sigma$ be the area of a variable element of Γ, regarded as an $(n-1)$-dimensional surface. Then the n-dimensional version of Green's theorem states that

$$\int_R \sum_{i=1}^n \frac{\partial}{\partial x_i}[F_{u_{x_i}}\phi(x)] dx = \int_\Gamma \phi(x)(G, \nu) d\sigma,$$

where $G = (F_{u_{x_1}}, \cdots, F_{u_{x_n}})$ is the n-dimensional vector whose components are the derivatives $F_{u_{x_i}}$, $\nu = (\nu_1, \cdots, \nu_n)$ is the unit outward normal to Γ, and (G, ν) denotes the scalar product of G and ν. Therefore

$$\delta J = \varepsilon \int_R \left(F_u - \sum_{i=1}^n \frac{\partial}{\partial x_i} F_{u_{x_i}} \right) \phi(x) dx + \varepsilon \int_\Gamma \phi(x)(G, \nu) d\sigma.$$

In order for the functional $J[u]$ to have an extremum, we must require that $\delta J = 0$ for all admissible $\phi(x)$, in particular, that $\delta J = 0$ for all admissible $\phi(x)$ which vanishes on the boundary Γ. For such functions, δJ reduces to

$$\delta J = \int_R \left(F_u - \sum_{i=1}^n \frac{\partial}{\partial x_i} F_{u_{x_i}} \right) \phi(x) dx,$$

and then, because of the arbitrariness of $\phi(x)$ inside R, $\delta J = 0$ implies that

$$F_u - \sum_{i=1}^n \frac{\partial}{\partial x_i} F_{u_{x_i}} = 0$$

for all $x \in R$. □

21.2.

Proof. We follow the line of reasoning in Gelfand and Folmin [4], §6. The problem can be formulated as follows: *Among all curves whose end points lie on two given vertical lines $x = a$ and $x = b$, find the curve for which the functional*

$$J[y] = \int_a^b F(x, y, y') dx$$

has an extremum.

We begin by calculating the variation δJ of $J[y]$. As before, δJ means the principle linear part of the increment

$$\Delta J = J[y + h] - J[y] = \int_a^b [F(x, y + h, y' + h') - F(x, y, y')] dx.$$

Using Taylor's theorem to expand the integrand, we obtain

$$\Delta J = \int_a^b (F_y h + F_{y'} h') dx + \cdots,$$

where the dots denote terms of order higher than 1 relative to h and h', and hence

$$\delta J = \int_a^b (F_y h + F_{y'} h') dx.$$

Here, unlike the fixed end point problem, $h(x)$ need no longer vanish at the points a and b, so that integration by parts now gives

$$\begin{aligned} \delta J &= \int_a^b \left(F_y - \frac{d}{dx} F_{y'} \right) h(x) dx + F_{y'} h(x)|_{x=a}^{x=b} \\ &= \int_a^b \left(F_y - \frac{d}{dx} F_{y'} \right) h(x) dx + F_{y'}|_{x=b} h(b) - F_{y'}|_{x=a} h(a). \end{aligned}$$

We first consider function $h(x)$ such that $h(a) = h(b) = 0$. The rationale is that if y^* is an extremal among all admissible function, then y^* must be an extremal among the smaller class of functions whose values at end points agree with those of y^*. Then as in the simplest variational problem, the condition $\delta J = 0$ implies that

$$F_y - \frac{d}{dx} F_{y'} = 0.$$

Therefore, in order for the curve $y = y(x)$ to be a solution of the variable end point problem, y must be an extremal, i.e. a solution of Euler's equation. But if y is an extremal, the integral in the above expression for δJ vanishes, and then the condition $\delta J = 0$ takes the form

$$F_{y'}|_{x=b}h(b) - F_{y'}|_{x=a}h(a) = 0,$$

from which it follows that

$$F_{y'}|_{x=a} = 0, F_{y'}|_{x=b} = 0,$$

since $h(x)$ is arbitrary. Thus, to solve the variable end point problem, we must first find a general integral of Euler's equation, and then use the *natural boundary conditions* $F_{y'}|_{x=a} = F_{y'}|_{x=b} = 0$ to determine the values of the arbitrary constants. □

21.2 The Rayleigh–Ritz method and its application to the Sturm–Liouville problem

The textbook's explanation of the Rayleigh-Ritz method is a bit ambiguous. We therefore give a summary of this method, as presented in Gelfand and Folmin [4], Chapter 8.

The idea of the Rayleigh-Ritz method consists of two parts. First, convert a differential equation to a variational problem, in that the solution of the differential equation is the extremal of a variational problem. Second, in a certain function space, use a set of complete functions to approximate the extremal, so that each approximation is reduced to a finite-dimensional optimization problem.

As an example, consider the Sturm-Liouville problem: Let $P = P(x) > 0$ and $Q = Q(x)$ be two given functions, where Q is continuous and P is continuously differentiable, and consider the Sturm-Liouville equation

$$-(Py')' + Qy = \lambda y,$$

subject to the boundary conditions $y(a) = y(b) = 0$. It's required to find the eigenfunctions and eigenvalues of the above boundary value problem.

The following result converts the above problem of solving a differential equation into a problem of finding variational extremal (Gelfand and Fomin [4], §12, Theorem 1; also see §41.1):

Theorem 3. *Given the functional $J[y] = \int_a^b F(x, y, y')dx$, let the admissible curves satisfy the conditions*

$$y(a) = A,\ y(b) = B,\ K[y] = \int_a^b G(x, y, y')dx = l,$$

where $K[y]$ is another functional, and let $J[y]$ have an extremum for $y = y(x)$. Then, if $y = y(x)$ is not an extremal of $K[y]$, there exists a constant λ such that $y = y(x)$ is an extremal of the functional $\int_a^b (F + \lambda G)dx$, i.e. $y = y(x)$ satisfies the differential equation

$$F_y - \frac{d}{dx}F_{y'} + \lambda\left(G_y - \frac{d}{dx}G_{y'}\right) = 0$$

With the above result, the Sturm-Liouville problem is reduced to finding an extremum of the quadratic functional

$$J[y] = \int_a^b (Py'^2 + Qy^2)dx,$$

subject to the boundary conditions $y(a) = y(b) = 0$ and the subsidiary condition $\int_a^b y^2 dx = 1$.[2]

Then we can apply the Rayleigh–Ritz method as follows. Suppose we are looking for the minimum of a functional $J[y]$ defined on some space $\mathcal{M}$ of admissible functions, which for simplicity we take to be a normed linear space. Let φ_1, φ_2, $\cdots$ be an infinite sequence of functions in $\mathcal{M}$, and let $\mathcal{M}_n$ be the n-dimensional linear subspace of $\mathcal{M}$ spanned by the first n of the functions $(\varphi_i)_{i=1}^{\infty}$. Then on each subspace $\mathcal{M}_n$, the functional $J[y]$ leads to a function $J[\alpha_1\varphi_1 + \cdots + \alpha_n\varphi_n]$ of the n variables $\alpha_1, \cdots, \alpha_n$.

[2] Use Theorem 3, changing λ to $-\lambda$.

Next, we choose $\alpha_1, \cdots, \alpha_n$ in such a way as to minimize $J[\alpha_1\varphi_1 + \cdots + \alpha_n\varphi_n]$, denoting the minimum by μ_n and the element of $\mathcal{M}_n$ which yields the minimum by y_n. We say the sequence $(\varphi_i)_{i=1}^\infty$ is *complete (in $\mathcal{M}$)* if given any $y \in \mathcal{M}$ and any $\varepsilon > 0$, there is a finite linear combination η_n of the form $\eta_n = \sum_{i=1}^n \alpha_i\varphi_i$ such that $\|\eta_n - y\| < \varepsilon$ (where n depends on ε). Then we have the following theorem

Theorem 4. *If the functional $J[y]$ is continuous in the norm of $\mathcal{M}$, and if the sequence $(\varphi_i)_{i=1}^\infty$ is complete, then*

$$\lim_{n\to\infty} \mu_n = \mu,$$

where $\mu = \inf_y J[y]$.

In the particular case of the Sturm–Liouville problem, we can choose $\varphi_k(x) = \sin(kx)$. Then $(\mu_n)_{n=1}^\infty$ converges to the smallest eigenvalue of the Sturm–Liouville equation and $(y_n)_{n=1}^\infty$ converges to the corresponding eigenfunction. We can continue this procedure to find the rest of the eigenvalues and eigenfunctions (see Gelfand and Fomin [4], §41.4). This is summarized in the following result

Theorem 5. *The Sturm-Liouville problem has an infinite sequence of eigenvalues $\lambda^{(1)}$, $\lambda^{(2)}$, $\cdots$, and to each eigenvalue $\lambda^{(n)}$ there corresponds an eigenfunction $y^{(n)}$ which is unique to within a constant factor.*

21.3 Solutions of the exercise problems from Gelfand and Fomin [4], Chapter 8

2.

Proof. We first calculate the exact solution. Let $F(x, y, y') = y'^2 - y^2 - 2xy$. Then Euler's equation becomes

$$F_y - \frac{d}{dx}F_{y'} = -2y - 2x - \frac{d}{dx}(2y') = -2(y + x + y'').$$

To solve the second order linear ODE

$$\begin{cases} y'' + y = -x \\ y(0) = y(1) = 0, \end{cases}$$

we can employ at least four methods: the Green's function for boundary value problems, Fourier expansion over the interval $(0, 1)$, operator calculus (including Laplace transform, see Ding and Li [2]), and reduction to a system of first order ODEs (also see Ding and Li [2]). However, it's very easy to see directly the general solution of $y'' + y = -x$ is $-x + C_1 \cos x + C_2 \sin x$. Using boundary condition, we conclude $y(x) = -x + \csc 1 \sin x$. In this case, it's easy to calculate $J[y] = \frac{1}{3}$.

We then use the Ritz method to find an approximate solution. For this purpose, we need to show the completeness of $\{\varphi_n\}_{n=1}^\infty$ in the space $\mathcal{M} = \{f \in \mathcal{D}_1(0, 1) : f(0) = f(1) = 0\}$, where $\varphi_n = x^n(1 - x)$ and the norm is that of $\mathcal{D}_1(0, 1)$ (i.e. $\|f\| = \max_{0\le x\le 1} |f(x)| + \max_{0\le x\le 1} |f'(x)|$).[3] We have the following lemmas.

Lemma 1. *The set of polynomials is dense in $\mathcal{D}_1(0, 1)$.*

Proof. By the Weierstrass approximation theorem, for any element $f \in \mathcal{D}_1(0, 1)$, we can find a sequence $(P_n)_{n=1}^\infty$ of polynomials, such that

$$\max_{0\le x\le 1} |f'(x) - P_n(x)| \le \frac{1}{n}.$$

Let $Q_n(x) = \int_0^x P_n(\xi)d\xi + f(0)$. Then Q_n is still a polynomial and

$$\begin{aligned} \|Q_n - f\| &= \max_{0\le x\le 1} \left| \int_0^x P_n(\xi)d\xi + f(0) - \left[\int_0^x f'(\xi)d\xi + f(0)\right] \right| + \max_{0\le x\le 1} |P_n(x) - f'(x)| \\ &\le \max_{0\le x\le 1} \int_0^x |P_n(\xi) - f'(\xi)|d\xi + \frac{1}{n} \\ &\le \frac{2}{n}. \end{aligned}$$

This proves the lemma. □

[3] For definition of the normed space $\mathcal{D}_n(a, b)$, see Gelfand and Fomin [4], page 7.

Lemma 2. *The linear space spanned by $\{\varphi_n\}_{n=1}^{\infty}$ is dense in $\mathcal{M}$ under the norm of $\mathcal{D}_1(0,1)$.*

Proof. It's clear each $\varphi_n \in \mathcal{M}$. Hence any finite linear combination of φ_n's belongs to $\mathcal{M}$. For any $f \in \mathcal{M}$, by Lemma 1, we can find $(\varepsilon_n)_{n=1}^{\infty}$ with $\varepsilon_n \downarrow 0$ and a sequence $(P_n)_{n=1}^{\infty}$ of polynomials such that

$$\|P_n - f\| \le \varepsilon_n.$$

Each P_n can be written in the form of $Q_n(x) + \alpha_n + \beta_n(1-x)$, where $Q_n(x)$ is a finite linear combination of φ_i's. Since $\|P_n - f\| \to 0$ as $n \to \infty$, we have $\alpha_n = P_n(1) \to f(1) = 0$ and $\alpha_n + \beta_n = P_n(0) \to f(0) = 0$ as $n \to \infty$. Therefore, we have by triangle inequality

$$\|Q_n - f\| \le \|\alpha_n + \beta_n(1-x)\| + \|P_n - f\| \le \varepsilon_n + |\alpha_n| + 2|\beta_n| \to 0$$

as $n \to \infty$. This shows the linear space spanned by $\{\varphi_n\}_{n=1}^{\infty}$ is dense in $\mathcal{M}$. □

Combining Lemma 1 and Lemma 2, we can by Theorem 4 find the approximate solution of the original variational problem. Indeed, consider the n-th degree approximate solution $y_n = \sum_{k=1}^{n} a_{nk}\varphi_k(x)$. For simplicity of notation, we write a_k for each a_{nk}. Then finding the extremum of $J[y_n] = \int_0^1 (y_n'^2 - y_n^2 - 2xy_n)dx$ becomes the minimization of a function of n variables $a_1, a_2, \cdots, a_n$:

$$\arg\min_{(a_1,\cdots,a_n)\in\mathbb{R}^n} J[y_n].$$

Note for $i = 1, \cdots, n$, we have

$$\begin{aligned}
\frac{\partial}{\partial a_i} J[y_n] &= \int_0^1 \frac{\partial}{\partial a_i}(y_n'^2 - y_n^2 - 2xy_n)dx \\
&= 2\int_0^1 \left(y_n' \frac{\partial}{\partial a_i} y_n' - y_n \frac{\partial}{\partial a_i} y_n - x \frac{\partial}{\partial a_i} y_n \right) dx \\
&= 2\int_0^1 \left(y_n'\varphi_i'(x) - y_n\varphi_i(x) - x\varphi_i(x) \right) dx \\
&= 2\left[\sum_{k=1}^{n} a_k \left(\frac{ki}{k+i-1} - \frac{2ki+i+k}{i+k} + \frac{ik+i+k}{k+i+1} + \frac{2}{k+i+2} - \frac{1}{k+i+3} \right) - \frac{1}{(i+2)(i+3)} \right].
\end{aligned}$$

So the extremal $y_n = \sum_{k=1}^{n} a_k\varphi_k(x)$ are given by solving the linear equation

$$A \begin{bmatrix} a_1 \\ a_2 \\ \cdots \\ a_n \end{bmatrix} = \begin{bmatrix} \frac{1}{3\cdot 4} \\ \frac{1}{4\cdot 5} \\ \cdots \\ \frac{1}{(n+2)(n+3)} \end{bmatrix},$$

where $A_{ik} = \frac{ki}{k+i-1} - \frac{2ki+i+k}{i+k} + \frac{ik+i+k}{k+i+1} + \frac{2}{k+i+2} - \frac{1}{k+i+3}$.

Therefore, to find the n-th degree approximation of the extremal y as well as the corresponding extreme $J[y]$, we first solve the above system of n linear equations to get y_n, and then apply either the quadrature method or an explicit formula to evaluate $J[y_n]$.

Remark 29. *Note the matrix A is ill-conditioned as n grows. So we postpone a numerical illustration of the above approximation to next version of the solution manual.*

□

8.

Proof. Since on a finite interval, L^2-convergence implies L^1-convergence (by the Cauchy-Schwarz inequality), we can without loss of generality assume the meaning of "in the mean" is in L^2-sense. By extending $h''(x)$ to an odd function on $[-\pi,\pi]$ (not necessarily continuous at 0), we can write $h''(x)$ on $[0,\pi]$ as

$$h''(x)=\sum_{r=1}^{\infty}A_r\sin(rx),$$

where $A_r=\frac{2}{\pi}\int_0^\pi h''(x)\sin(rx)dx$. Define

$$h_n(x)=-\sum_{r=1}^{n}\frac{A_r}{r^2}\sin(rx).$$

Then $h_n''(x)\to h''(x)$ in $L^2(0,\pi)$ as $n\to\infty$.

Meanwhile, we have (note $h'(0)=0$)

$$\begin{aligned}|h_n'(x)-h'(x)| &= \left|\int_0^x h_n''(\xi)d\xi+h_n'(0)-\int_0^x h''(\xi)d\xi-h'(0)\right|\\ &\le \int_0^x |h_n''(\xi)-h''(\xi)|d\xi+|h_n'(0)|\\ &\le \|h_n''-h''\|_{L^2(0,\pi)}\sqrt{\pi}+|h_n'(0)|.\end{aligned}$$

So $\|h_n'-h'\|_{L^2(0,\pi)}\le\|h_n''-h''\|_{L^2(0,\pi)}\pi+|h_n'(0)|\sqrt{\pi}$. We further note

$$h_n'(0)=-\sum_{r=1}^{n}\frac{A_r}{r}=-\sum_{r=1}^{n}\frac{2}{r\pi}\int_0^\pi h''(x)\sin(rx)dx=\sum_{r=1}^{n}\frac{2}{\pi}\int_0^\pi h'(x)\cos(rx)dx.$$

By extending $h'(x)$ to a continuous even function on $[-\pi,\pi]$, we can conclude that the Fourier cosine expansion of $h'(x)$ (note $h(0)=h(\pi)=0$)

$$\sum_{r=0}^{n}\frac{2}{\pi}\int_0^\pi h'(x)\cos(rx)dx\cdot\cos(rx)=\sum_{r=1}^{n}\frac{2}{\pi}\int_0^\pi h'(x)\cos(rx)dx\cdot\cos(rx)$$

converges to $h'(x)$ pointwise as $n\to\infty$. In particular, $h_n'(0)=\sum_{r=1}^n\frac{2}{\pi}\int_0^\pi h'(x)\cos(rx)dx\cdot\cos(0\cdot x)\to h'(0)=0$ as $n\to\infty$. Combining with the inequality $\|h_n'-h'\|_{L^2(0,\pi)}\le\|h_n''-h''\|_{L^2(0,\pi)}\pi+|h_n'(0)|\sqrt{\pi}$, we conclude $h_n'(x)\to h'(x)$ in $L^2(0,\pi)$ as $n\to\infty$.

Finally, we have

$$|h_n(x)-h(x)|=\left|\int_0^x h_n'(\xi)d\xi+h_n(0)-\int_0^x h'(\xi)d\xi-h(0)\right|\le\int_0^x|h_n'(\xi)-h'(\xi)|d\xi\le\|h_n'-h'\|_{L^2(0,\pi)}\sqrt{\pi}.$$

So $\|h_n-h\|_{L^2(0,\pi)}\le\|h_n'-h'\|_{L^2(0,\pi)}\pi\to0$ as $n\to\infty$.

In summary, we conclude as $n\to\infty$, $h_n\to h$, $h_n'\to h'$, and $h_n''\to h''$ all in $L^2(0,\pi)$. And $C_r^n=-\frac{A_r}{r^2}=-\frac{2}{r^2\pi}\int_0^\pi h''(x)\sin(rx)dx$ is clearly independent of n. □

9.

Proof.

$$\begin{aligned}\left|\int_a^b f_n(x)g_n(x)dx-\int_a^b f(x)g(x)dx\right| &\le \int_a^b|f_n(x)||g_n(x)-g(x)|dx+\int_a^b|g(x)||f_n(x)-f(x)|dx\\ &\le \max_{a\le x\le b}|g_n(x)-g(x)|\|f_n\|_{L^2(a,b)}\sqrt{b-a}+\|f_n-f\|_{L^2(a,b)}\|g\|^2_{L^2(a,b)}.\end{aligned}$$

Since $f_n\to f$ in mean, $(\|f_n\|_{L^2(a,b)})_{n=1}^\infty$ is bounded. As $n\to\infty$, the RHS of the above inequality goes to 0, so we can conclude $\int_a^b f_n(x)g_n(x)dx\to\int_a^b f(x)g(x)dx$. □

21.4 Exercise at the end of chapter

1. (1)

Proof. $F(x,y,y')=\sqrt{1+y^2y'^2}$. So the Euler-Lagrange equation becomes

$$\begin{aligned}\frac{\partial F}{\partial y}-\frac{d}{dx}\frac{\partial F}{\partial y'} &= \frac{yy'^2}{\sqrt{1+y^2y'^2}}-\frac{d}{dx}\frac{y'y^2}{\sqrt{1+y^2y'^2}}\\ &= \frac{yy'^2}{\sqrt{1+y^2y'^2}}-\frac{(y''y^2+2yy'^2)\sqrt{1+y^2y'^2}-y'y^2\frac{2yy'^3+2y^2y'y''}{2\sqrt{1+y^2y'^2}}}{1+y^2y'^2}\\ &= 0.\end{aligned}$$

After simplification, we have $0=y''y+y'^2=(y'y)'=\left(\frac{1}{2}y^2\right)''$. Therefore, the general solution is $y^2=ax+b$. □

(2)

Proof. $F(x,y,y')=y^2+y'^2$, so the Euler-Lagrange equation becomes

$$\frac{\partial F}{\partial y}-\frac{d}{dx}\frac{\partial F}{\partial y'}=2y-\frac{d}{dx}(2y')=2y-2y''=0.$$

Therefore $y=ae^x+be^{-x}$. □

(3)

Proof. $F(x,y,y')=\frac{x}{x+y'}$. So the Euler-Lagrange equation becomes

$$\frac{\partial F}{\partial y}-\frac{d}{dx}\frac{\partial F}{\partial y'}=\frac{d}{dx}\frac{x}{(x+y')^2}=0.$$

So there exists some constant C such that $\frac{x}{(x+y')^2}=C$, which gives $y'=-x\pm\sqrt{\frac{x}{C}}$. Use a change-of-variable, we have $y'=\frac{3a}{2}x^{1/2}-x$. Hence $y=ax^{3/2}-\frac{x^2}{2}+b$. □

(4)

Proof. $F(x,y,y')=\sqrt{1+x}\sqrt{1+y'^2}$. So the Euler-Lagrange equation becomes

$$0-\frac{d}{dx}\left[\sqrt{1+x}\frac{2y'}{2\sqrt{1+y'^2}}\right]=0.$$

There must exist a constant a such that $\sqrt{1+x}\frac{y'}{\sqrt{1+y'^2}}=a$. Solving this equation gives $y'=\frac{a}{\sqrt{1+x-a^2}}$. Therefore $y=2a\sqrt{1+x-a^2}+b$. □

2.

Proof. The point on the cone $x^2+y^2=z^2$ can be described by the parametric coordinate $(z\cos\theta,z\sin\theta,z)$ $(0\le\theta<2\pi)$. For any given curve γ on the cone, assuming γ is parametrized by θ, the length of γ is

$$\int_{(x_0,y_0)}^{(x_1,y_1)}\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2+\left(\frac{dz}{d\theta}\right)^2}d\theta=\int_{(x_0,y_0)}^{(x_1,y_1)}\sqrt{z^2+2(z')^2}d\theta=\int_{(x_0,y_0)}^{(x_1,y_1)}F(\theta,z,z')d\theta,$$

where $F(\theta, z, z') = \sqrt{z^2 + 2(z')^2}$. So the Euler-Lagrange equation becomes

$$\begin{aligned}
\frac{\partial F}{\partial z} - \frac{d}{d\theta}\frac{\partial F}{\partial z'} &= \frac{z}{\sqrt{z^2+2(z')^2}} - \frac{d}{d\theta}\frac{2z'}{\sqrt{z^2+2(z')^2}} \\
&= \frac{z}{\sqrt{z^2+2(z')^2}} - \frac{2z''\sqrt{z^2+2(z')^2} - 2z'\frac{zz'+2z'z''}{\sqrt{z^2+2(z')^2}}}{z^2+2(z')^2} \\
&= \frac{z}{\sqrt{z^2+2(z')^2}} - \frac{2z''[z^2+2(z')^2] - 2(z')^2(z+2z'')}{[z^2+2(z')^2]\sqrt{z^2+2(z')^2}} \\
&= 0.
\end{aligned}$$

Simplifying the above equation, we have

$$z[z^2 + 2(z')^2] - 2z''z^2 - 4(z')^2z'' + 2(z')^2z + 4(z')^2z'' = z[z^2 + 4(z')^2 - 2zz''] = 0.$$

So a non-trivial solution must satisfy the equation $z'' - \frac{2}{z}(z')^2 - \frac{z}{2} = 0$ or equivalently $\frac{z''}{z} - \frac{2(z')^2}{z^2} = \frac{1}{2}$. Note $\left(\frac{z'}{z}\right)' = \frac{z''z-(z')^2}{z^2} = \frac{z''}{z} - \left(\frac{z'}{z}\right)^2$. Using the substitution $\frac{z''}{z} = \left(\frac{z'}{z}\right)' + \left(\frac{z'}{z}\right)^2$, we transform the original equation to

$$\left(\frac{z'}{z}\right)' - \left(\frac{z'}{z}\right)^2 = \frac{1}{2}.$$

Define $h(\theta) = \frac{z'(\theta)}{z(\theta)} = \frac{z'}{z}$, we get a Riccati equation of h:

$$h' = \frac{1}{2} + h^2.$$

It is well-known that a general Riccati equation can always be reduced to a second order linear ordinary differential equation (see Remark 30 below). For our particular case, we use the substitution $h = -\frac{w'}{w}$, then

$$\frac{1}{2} = h' - h^2 = -\frac{w''}{w} + \frac{(w')^2}{w^2} - \left(\frac{w'}{w}\right)^2 = -\frac{w''}{w}.$$

So w satisfies the equation $w'' + \frac{1}{2}w = 0$, which has a general solution $w = C_1 \cos\frac{\theta}{\sqrt{2}} + C_2 \sin\frac{\theta}{\sqrt{2}}$. By using trigonometric identities, we can write w as $\frac{1}{a}\cos\frac{\theta+b}{\sqrt{2}}$. Note $-\frac{w'}{w} = h = \frac{z'}{z}$, we conclude $z(\theta) = \frac{1}{w(\theta)} = a\sec\frac{\theta+b}{\sqrt{2}}$.

Remark 30. *In mathematics, a Riccati equation is any ordinary differential equation that has the form $y' = q_0(x) + q_1(x)y + q_2(x)y^2$ where $q_0(x) \neq 0$ and $q_2(x) \neq 0$. It can be reduced to a second order linear equation by the following procedure (see, for example, wikipedia). First, use the substitution $v = q_2(x)y$. Then the original equation becomes*

$$v' = v^2 + R(x)v + S(x),$$

where $S(x) = q_0(x)q_2(x)$ and $R(x) = q_1(x) + q_2'(x)/q_2(x)$. Then substitute $v = -u'/u$ and it follows that u satisfies the linear second order ODE

$$u'' - R(x)u' + S(x)u = 0.$$

A solution of this equation will lead to a solution $y = -u'/(uq_2(x))$ of the original Riccati equation.

□

3.

Proof. Points on the cylindrical surface are described by the parametric coordinates $(r\cos\theta, r\sin\theta, z)$ where $r > 0$ is a constant and $\theta \in [0, 2\pi)$. For any curve γ on the cylindrical surface, assuming γ is parametrized by θ, then the length of γ is

$$\int_{(x_0,y_0)}^{(x_1,y_1)} \sqrt{r^2 + (z')^2}d\theta = \int_{(x_0,y_0)}^{(x_1,y_1)} F(\theta, z, z')d\theta,$$

where $F(\theta, z, z') = \sqrt{r^2 + (z')^2}$ and the Euler-Lagrange equation becomes

$$\frac{\partial F}{\partial z} - \frac{d}{d\theta}\frac{\partial F}{\partial z'} = -\frac{d}{d\theta}\frac{z'}{\sqrt{r^2 + (z')^2}} = -\frac{z''\sqrt{r^2 + (z')^2} + z'\frac{z'z''}{\sqrt{r^2+(z')^2}}}{r^2 + (z')^2}.$$

After simplification, we get $z''[r^2 + 2(z')^2] = 0$ and hence $z'' = 0$. This implies $z(\theta) = a + b\theta$ for some constants a and b. □

22 Overview of Equations of Mathematical Physics

22.1 Summary on the classification of second order linear PDE

Suppose we have a general second order linear partial differential equation

$$\sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=0}^{n} B_i \frac{\partial u}{\partial x_i} + Cu = F.$$

If we choose a non-singular change of variable

$$\begin{cases} x_1 = x_1(y_1, \cdots, y_n) \\ x_2 = x_2(y_1, \cdots, y_n) \\ \cdots \\ x_n = x_n(y_1, \cdots, y_n), \end{cases}$$

then the original equation becomese

$$\sum_{k=1}^{n}\sum_{l=1}^{n} \bar{A}_{kl}\frac{\partial^2 u}{\partial y_k \partial y_l} + \sum_{l=1}^{n} \bar{B}_l \frac{\partial u}{\partial y_l} + Cu = F,$$

with

$$\bar{A}_{kl} = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}\frac{\partial y_k}{\partial x_i}\frac{\partial y_l}{\partial x_j}, \ \bar{B}_l = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}\frac{\partial^2 y_l}{\partial x_i \partial x_j} + \sum_{i=1}^{n} B_i\frac{\partial y_l}{\partial x_i}.$$

In matrix form, we have

$$\bar{A} = (\bar{A}_{kl}) = JAJ^T,$$

where

$$J = \frac{\partial(y_1, \cdots, y_n)}{\partial(x_1, \cdots, x_n)} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial y_n}{\partial x_1} & \frac{\partial y_n}{\partial x_2} & \cdots & \frac{\partial y_n}{\partial x_n} \end{bmatrix}.$$

Since A is a real symmetric matrix, we can find an orthogonal matrix $U(= U(x_1, x_2, \cdots, x_n))$ such that UAU^T is a diagonal matrix. Then we obtain n^2 equations $\frac{\partial y_i}{\partial x_j} = u_{ij}(x_1, \cdots, x_n)$ $(i, j = 1, \cdots, n)$.

In the special case of $n = 2$, we suppose the second order differential operator in the original equation is

$$Lu \equiv A\frac{\partial^2 u}{\partial x \partial y} + 2B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2}.$$

Then with the change of variable $\xi = \xi(x, y)$, $\eta = \eta(x, y)$, the operator Lu takes the form

$$Lu = A_1\frac{\partial^2 u}{\partial \xi^2} + 2B_1\frac{\partial^2 u}{\partial \xi \partial \eta} + C_1\frac{\partial^2 u}{\partial \eta^2} + \frac{\partial u}{\partial \xi}L\xi + \frac{\partial u}{\partial \eta}L\eta,$$

where

$$A_1 = A\left(\frac{\partial \xi}{\partial x}\right)^2 + 2B\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial \eta} + C\left(\frac{\partial \xi}{\partial y}\right)^2,$$

$$B_1 = A\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x} + B\left(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial \eta}{\partial x}\frac{\partial \xi}{\partial y}\right) + C\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y},$$

$$C_1 = A\left(\frac{\partial \eta}{\partial x}\right)^2 + 2B\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial \eta} + C\left(\frac{\partial \eta}{\partial y}\right)^2.$$

As a consequence, we have $B_1^2 - A_1C_1 = \left|\frac{\partial(\xi,\eta)}{\partial(x,y)}\right|^2 (B^2 - AC)$. Beside Theorem 22.1 and formulas (22.7)-(22.13) in the textbook, we also have the following more explicit results.

1) $B^2 = AC$. In this case, put $k = \frac{B}{A} = \frac{C}{B}$ and we have

$$A_1 = A\left(\frac{\partial \xi}{\partial x} + k\frac{\partial \xi}{\partial y}\right)^2, \ B_1 = A\left(\frac{\partial \xi}{\partial x} + k\frac{\partial \xi}{\partial y}\right)\left(\frac{\partial \eta}{\partial x} + k\frac{\partial \eta}{\partial y}\right), \ C_1 = A\left(\frac{\partial \eta}{\partial x} + k\frac{\partial \eta}{\partial y}\right)^2.$$

In order that both B_1 and C_1 vanish it is sufficient to put

$$\frac{\partial \eta}{\partial x} + k\frac{\partial \eta}{\partial y} = 0.$$

For the solution of first order linear PDE, see Ding and Li [2]. ξ can be chosen arbitrarily in this case, as far as the change of variable $(x, y) \mapsto (\xi, \eta)$ is non-singular.

2) $B^2 > AC$. First assume A does not vanish (the case $C \neq 0$, $A = 0$ can be treated in a similar way; the case $A = C = 0$ we shall deal with separately). We put $\xi = x$, $\eta = \varphi(x, y)$. Then the condition that B_1 should vanish becomes

$$A\frac{\partial \eta}{\partial x} + B\frac{\partial \eta}{\partial y} = 0.$$

Solving for η, we can obtained the canonical form of a hyperbolic second order linear PDE. If $A = C = 0$, then the original equation will have a principle term of the form $\frac{\partial^2 u}{\partial x \partial y}$. Use the substitution $\xi = x + y$, $\eta = x - y$, the equation takes the canonical form

$$\frac{\partial^2 u}{\partial xi^2} - \frac{\partial^2 u}{\partial \eta^2} + \cdots.$$

22.2 Exercise at the end of chapter

1. (1)

Proof. Using the notation of Theorem 22.1, we have $a = 1$, $b = 0$, $c = y$. If $b^2 - ac > 0$, i.e. $y < 0$, the ODE for characteristics is

$$\frac{dy}{dx} = \pm\sqrt{-y}.$$

So the two integral curves are $\sqrt{-y} - \frac{x}{2} = C_1$ and $\sqrt{-y} + \frac{x}{2} = C_2$, or equivalently, $x = C_1$ and $2\sqrt{-y} = C_2$. Under the change of variable $\xi = x$, $\eta = 2\sqrt{-y}$, the equation is simplified to $\frac{\partial^2 u}{\partial \xi^2} - \frac{\partial^2 u}{\partial \eta^2} = 0$. If $b^2 - ac < 0$, i.e. $y > 0$, the ODE for characteristics is

$$\frac{dy}{dx} = \pm i\sqrt{y}.$$

By similar argument, we should choose the change of variable $\xi = x$, $\eta = 2\sqrt{y}$. Then the original equation is reduced to $\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} = 0$. □

(2)

Proof. Using the notation of Theorem 22.1, we have $a = 1 + x^2$, $b = 0$, and $c = 1 + y^2$. So the ODE for characteristics is

$$\frac{dy}{dx} = \pm i\sqrt{(1+x^2)(1+y^2)}.$$

Writing it in the form of separated variables, we have

$$\frac{dy}{\sqrt{1+y^2}} = \pm i\frac{dx}{\sqrt{1+x^2}}.$$

We can solve these equations to obtain the change of variable

$$\begin{cases} \xi = \text{arcsinh} x \\ \eta = \text{arcsinh} y, \end{cases}$$

under which the original equation is reduced to $\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} = 0$. □

(3)

Proof. Using the notation of Theorem 22.1, we have $a = \tan^2 x$, $b = -y\tan x$, and $c = y^2$. Then $\Delta = b^2 - ac = 0$. So the ODE for characteristics is

$$\frac{dy}{dx} = -\frac{y}{\tan x},$$

which has the general solution $y = C\sin x$. Let $\xi = y\sin x$ and $\eta = y\cos x$. The original equation is simplified to

$$(\xi^2+\eta^2)\frac{\partial^2 u}{\partial \eta^2} - \xi\frac{\partial u}{\partial \xi} + \eta\frac{\partial u}{\partial \eta} = 0.$$

□

(4)

Proof. Using the notation of Theorem 22.1, we have $a = 1$, $b = -\sin x$, $c = -\cos^2 x$. So the ODE for characteristic becomes

$$\frac{dy}{dx} = -\sin x,$$

which has a general solution $y = \cos x + C$. Calculation shows the following change of variable simplifies the original equation most

$$\begin{cases} \xi = x + y - \cos x \\ \eta = x - y + \cos x, \end{cases}$$

under which the original equation becomes $\frac{\partial^2 u}{\partial \xi \partial \eta} = 0$. □

2. (1)

Proof.

$$\frac{\partial u}{\partial x} = e^{-(ax+by)}\left[-av(x,y) + \frac{\partial v}{\partial x}\right], \ \frac{\partial^2 u}{\partial x^2} = e^{-(ax+by)}\left\{-a\left[-av + \frac{\partial v}{\partial x}\right] - a\frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial x^2}\right\}.$$

By symmetry, we have

$$\frac{\partial u}{\partial y} = e^{-(ax+by)}\left[-bv(x,y) + \frac{\partial v}{\partial y}\right], \ \frac{\partial^2 u}{\partial y^2} = e^{-(ax+by)}\left\{-b\left[-bv + \frac{\partial v}{\partial y}\right] - b\frac{\partial v}{\partial y} + \frac{\partial^2 v}{\partial y^2}\right\}.$$

Therefore

$$
\begin{aligned}
& \bigtriangledown^2 u + 2a\frac{\partial u}{\partial x} + 2b\frac{\partial u}{\partial y} \\
= \;& e^{-(ax+by)}\left[(a^2+b^2)v - 2a\frac{\partial v}{\partial x} - 2b\frac{\partial v}{\partial y} + \bigtriangledown^2 v - 2(a^2+b^2)v + 2a\frac{\partial v}{\partial x} + 2b\frac{\partial v}{\partial y}\right] \\
= \;& e^{-(ax+by)}\left[\bigtriangledown^2 v - (a^2+b^2)v\right].
\end{aligned}
$$

So the original equation is transformed into $\bigtriangledown^2 v - (a^2+b^2)v = 0$. □

(2)

Proof. Follow the hint and use the transformation $u(x,y) = e^{-ax+by}v(x,y)$. □

(3)

Proof. Let $u(x,y,t) = v(x,y,t)h(\alpha,\beta,\gamma,x,y,t)$, where $h(\alpha,\beta,\gamma,x,y,t) = e^{\alpha x+\beta y+\gamma t}$. Then it's easy to see

$$
\frac{\partial u}{\partial x} = h\left(\alpha v + \frac{\partial v}{\partial x}\right), \; \frac{\partial u}{\partial y} = h\left(\beta v + \frac{\partial v}{\partial y}\right), \; \frac{\partial u}{\partial t} = h\left(\gamma v + \frac{\partial v}{\partial t}\right),
$$

and

$$
\frac{\partial^2 u}{\partial x^2} = h\left[\alpha\left(\alpha v + \frac{\partial v}{\partial x}\right) + \alpha\frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial x^2}\right], \; \frac{\partial^2 u}{\partial y^2} = h\left[\beta\left(\beta v + \frac{\partial v}{\partial y}\right) + \beta\frac{\partial v}{\partial y} + \frac{\partial^2 v}{\partial y^2}\right],
$$

$$
\frac{\partial^2 u}{\partial x \partial y} = h\left[\beta\left(\alpha v + \frac{\partial v}{\partial x}\right) + \alpha\frac{\partial v}{\partial y} + \frac{\partial^2 v}{\partial x \partial y}\right].
$$

Therefore

$$
\begin{aligned}
& h^{-1}\left[a\frac{\partial^2 u}{\partial x^2} + 2b\frac{\partial^2 u}{\partial x\partial y} + c\frac{\partial^2 u}{\partial y^2} + d\frac{\partial u}{\partial x} + e\frac{\partial u}{\partial y} + fu - \frac{\partial u}{\partial t}\right] \\
= \;& (a\alpha^2 + c\beta^2 + 2b\alpha\beta + d\alpha + e\beta)v + (2a\alpha + 2b\beta + d)\frac{\partial v}{\partial x} + (2c\beta + 2b\alpha + e)\frac{\partial v}{\partial y} + (f-\gamma)v \\
& + a\frac{\partial^2 v}{\partial x^2} + 2b\frac{\partial^2 v}{\partial x\partial y} + c\frac{\partial^2 v}{\partial y^2} - \frac{\partial v}{\partial t}.
\end{aligned}
$$

Solving the equation

$$
2\begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = -\begin{bmatrix} d \\ e \end{bmatrix},
$$

we have $\alpha = \frac{be-cd}{2(ac-b^2)}$, $\beta = \frac{bd-ae}{2(ac-b^2)}$. Note

$$
a\alpha^2 + c\beta^2 + 2b\alpha\beta + d\alpha + e\beta = [\alpha,\beta]\begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} + [\alpha,\beta]\begin{bmatrix} d \\ e \end{bmatrix} = [\alpha,\beta]\left\{\begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} + \begin{bmatrix} d \\ e \end{bmatrix}\right\} = 0.
$$

So by choosing the above α and β, and by setting $\gamma = f$, we can transform the original equation into the form

$$
a\frac{\partial^2 v}{\partial x^2} + 2b\frac{\partial^2 v}{\partial x\partial y} + c\frac{\partial^2 v}{\partial y^2} = \frac{\partial v}{\partial t}.
$$

Remark 31. *Something wrong in the calculation? Check!*

□

3.

Proof. By Example 13.1 (p172), the general solution of

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = 0$$

can be written as $f(x+at)+g(x-at)$. Using the boundary condition, it's easy to see $f(x) = \phi\left(\frac{x}{2}\right) - g(0)$ and $g(x) = \psi\left(\frac{x}{2}\right) - f(0)$. So the solution must be

$$\phi\left(\frac{x+at}{2}\right) + \psi\left(\frac{x-at}{2}\right) - \phi(0).$$

□

4. (1)

Proof.

$$\frac{\partial u}{\partial x} = -e^{-x}\sin y,\ \frac{\partial^2 u}{\partial x^2} = e^{-x}\sin y,\ \frac{\partial u}{\partial y} = e^{-x}\cos y,\ \frac{\partial^2 u}{\partial y^2} = -e^{-x}\sin y.$$

So $\bigtriangledown^2 u = 0$. $u|_{y=0} = x$ is obvious. □

(2)

Proof. Let (x,y) approach $(0,1)$ along the y-axis, we have

$$u(x,y) = u(0,y) = \frac{1}{1-y^2} \to \infty.$$

So as along as $u(x,y)$ has a definite value at $(0,1)$, it cannot be continuous at $(0,1)$. The case of $(0,-1)$ can be proved similarly. □

A The Black-Scholes partial differential equation

In mathematical finance, the following PDE appears in the derivation of the Black-Scholes call option pricing formula ($K > 0$):

$$\begin{cases} c_t(t,x) + rxc_x(t,x) + \frac{1}{2}\sigma^2 x^2 c_{xx}(t,x) = rc(t,x),\ t\in[0,T),\ x\geq 0 \\ c(T,x) = (x-K)^+, \\ c(t,0) = 0,\ t\in[0,T], \\ \lim_{x\to\infty}[c(t,x) - (x - e^{-r(T-t)}K)] = 0,\ t\in[0,T]. \end{cases}$$

Many more PDEs important in mathematical finance can be found in Kohn [6]. For now, we focus on the explicit solution of the Black-Scholes PDE using the techniques from the current textbook. "*How do you derive and solve the Black-Scholes PDE*" is a frequently asked question in Wall Street job interviews. The presentation below is essentially that of Wilmott et al. [12], §5.4.

A.1 Derivation of the Black-Scholes PDE and its boundary conditions

For the derivation of the main PDE, we refer to Shreve [10], §4.5.3 (see also Remark 32); for the derivation of the boundary conditions, we refer to Shreve [10], §4.5.4.

A.2 Simplification of the Black-Scholes PDE via change-of-variable

The first step of simplification is to convert $x\frac{\partial}{\partial x}$ and $x^2\frac{\partial^2}{\partial x^2}$ to $\frac{\partial}{\partial \eta}$ and $\frac{\partial^2}{\partial \eta^2}$, respectively, via some change of variable $x = f(\eta)$. This is the trick presented in §13.4 of the textbook. With the change-of-variable $x = e^\eta$ and $w(t,\eta) = c(t, e^\eta)$, we have

$$w_\eta(t,\eta) = xc_x(t,x),\ w_{\eta\eta}(t,\eta) = xc_x(t,x) + x^2c_{xx}(t,x).$$

So the original PDE becomes

$$w_t + \left(r - \frac{1}{2}\sigma^2\right)w_\eta + \frac{1}{2}\sigma^2 w_{\eta\eta} = rw,\ -\infty < \eta < \infty,\ t \in [0,T).$$

Or equivalently,

$$w_{\eta\eta} + (k-1)w_\eta + \frac{1}{\frac{1}{2}\sigma^2}w_t = kw,$$

where $k = r/\frac{1}{2}\sigma^2$.

The second simplification is to convert the equation into a form as close as possible to the standard heat equation

$$\frac{\partial v}{\partial \tau} - \alpha\Delta v = [...],$$

which means we need to normalize the coefficient of w_t to -1. Define $v(\tau,\eta) = v(\tau(t),\eta) = w(t,\eta)$. Then

$$\begin{aligned} & w_{\eta\eta}(t,\eta) + (k-1)w_\eta(t,\eta) + \frac{1}{\frac{1}{2}\sigma^2}w_t(t,\eta) - kw(t,\eta) \\ = \ & v_{\eta\eta}(\tau,\eta) + (k-1)v_\eta(\tau,\eta) + \frac{1}{\frac{1}{2}\sigma^2}\frac{d\tau}{dt}v_\tau(\tau,\eta) - kv(\tau,\eta). \end{aligned}$$

So we want to set $\tau = \frac{1}{2}\sigma^2(T-t)$. Then $v(\tau,\eta)$ satisfies the following PDE:

$$v_\tau = v_{\eta\eta} + (k-1)v_\eta - kv,\ -\infty < \eta < \infty,\ 0 < \tau \le \frac{1}{2}\sigma^2 T.$$

The third step of simplification is to remove the first order differential operator $\frac{\partial}{\partial \eta}$ so that we have the standard form of heat equation. To do so, we use the trick introduced in Exercise Problem 2 of Chapter 22 of the textbook. More precisely, rewriting the PDE for v in the following form

$$v_\tau + kv = v_{\eta\eta} + (k-1)v_\eta,$$

and motivated by the exponential integrating factor employed in solving first order ODE, we try the function $u(\tau,\eta) = e^{\alpha\eta+\beta\tau}v(\tau,\eta)$. Then

$$u_\eta = e^{\alpha\eta+\beta\tau}(\alpha v + v_\eta),\ u_{\eta\eta} = e^{\alpha\eta+\beta\tau}(\alpha^2 v + \alpha v_\eta + \alpha v_\eta + v_{\eta\eta}),\ u_\tau = e^{\alpha\eta+\beta\tau}(\beta v + v_\tau).$$

So we have

$$u_\tau - u_{\eta\eta} = e^{\alpha\eta+\beta t}[v_\tau + (\beta - \alpha^2)v - 2\alpha v_\eta - v_{\eta\eta}].$$

Comparing with the PDE for v: $v_\tau + kv - (k-1)v_\eta - v_{\eta\eta} = 0$, we want to set $\alpha = \frac{k-1}{2}$ and $\beta = \alpha^2 + k = \frac{1}{4}(k+1)^2$.

In summary, u satisfies the PDE

$$\begin{cases} u_\tau(\tau,\eta) = u_{\eta\eta}(\tau,\eta),\ -\infty < \eta < \infty,\ 0 < \tau \le \frac{1}{2}\sigma^2 T, \\ u(0,\eta) = \left(e^{\frac{k+1}{2}\eta} - Ke^{\frac{k-1}{2}\eta}\right)^+, \\ \lim_{\eta\to-\infty} u(\tau,\eta) = 0,\ \tau \in [0, \frac{1}{2}\sigma^2 T], \\ \lim_{\eta\to\infty}\left[e^{-\frac{k-1}{2}\eta - \frac{(k+1)^2}{4}\tau}u(\tau,\eta) - (e^\eta - Ke^{-rT/\frac{1}{2}\sigma^2})\right] = 0,\ \tau \in [0, \frac{1}{2}\sigma^2 T]. \end{cases}$$

and u is related to $c(t,x)$ in the following way

$$u(\tau,\eta)=e^{\frac{k-1}{2}\eta+\frac{(k+1)^2}{4}\tau}v(\tau,\eta)=e^{\frac{k-1}{2}\eta+\frac{(k+1)^2}{4}\tau}w\left(T-\frac{\tau}{\frac{1}{2}\sigma^2},\eta\right)=e^{\frac{k-1}{2}\eta+\frac{(k+1)^2}{4}\tau}c\left(T-\frac{\tau}{\frac{1}{2}\sigma^2},e^{\eta}\right),$$

or equivalently

$$c(t,x)=u\left(\frac{1}{2}\sigma^2(T-t),\ln x\right)e^{-\frac{k-1}{2}\ln x-\frac{(k+1)^2}{4}\frac{1}{2}\sigma^2(T-t)}.$$

Remark 32. *We have used change-of-variable throughout to simplify the Black-Scholes PDE. There is an observation that can simplify the PDE to begin with. Recall during the derivation of the Black-Scholes PDE, we used the fact that under the risk-neutral measure, the discounted call option price $e^{-rt}c(S_t,t)$ should be a martingale, where the underlying asset price process S_t satisfies the SDE $dS_t=rS_tdt+\sigma S_td\widetilde{W}_t$ ($\widetilde{W}_t$ is a Brownian motion under risk-neutral measure). Note $X_t=e^{-rt}S_t$ is a martingale satisfying the SDE $dX_t=\sigma X_td\widetilde{W}_t$, so instead of writing c as a function of S_t and t, we suppose c is a function of X_t and t. Then*

$$d[e^{-rt}c(X_t,t)]=e^{-rt}\left[-rc+c_t+\frac{1}{2}\sigma^2x^2c_{xx}\right]dt+\textit{martingale part}.$$

So the PDE has a simpler form: $c_t+\frac{1}{2}\sigma^2x^2c_{xx}=rc$. To remove x^2, we still need to set $x=e^{\eta}$ and $x^2\frac{\partial^2}{\partial x^2}$ becomes $\frac{\partial^2}{\partial\eta^2}-\frac{\partial}{\partial\eta}$. The resulting expression is not really as simple as we hoped, since first order differential operator still persists. But the coefficients become simpler. The rest of the simplification should proceed as before.

A.3 Solution of the simplified PDE and the Black-Scholes call option pricing formula

There are many methods to solve the initial value problem of heat equation over an infinite line. For example, we could use Fourier's transform. However, due to the messy form of the boundary value function, which is probably not easy for inverse Fourier transform, we employ the method of Green's function. Recall the fundamental solution of the equation $u_t=u_{\eta\eta}$ is

$$\frac{1}{2\sqrt{\pi t}}e^{-\frac{x^2}{4\tau}}.$$

So $u(x,\tau)$ can be obtained through the convolution formula:

$$\begin{aligned}u(\tau,\eta)&=\frac{1}{2\sqrt{\pi\tau}}\int_{-\infty}^{\infty}u(0,x)e^{-\frac{(\eta-x)^2}{4\tau}}dx\\&=\frac{1}{2\sqrt{\pi\tau}}\int_{-\infty}^{\infty}\left(e^{\frac{k+1}{2}\eta}-Ke^{\frac{k-1}{2}\eta}\right)^+e^{-\frac{(\eta-x)^2}{4\tau}}dx.\end{aligned}$$

After some tedious calculation, we can get the Black-Scholes call option pricing formula

$$c(S_t,t)=S_tN(d_1)-Ke^{-r(T-t)}N(d_2),$$

where

$$d_1=\frac{\log\frac{S_t}{K}+\left(r+\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}},\ d_2=\frac{\log\frac{S_t}{K}+\left(r-\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}.$$

Remark 33. *An alternative, and much easier, method, is via the Feymann-Kac formula, see Øksendal [7] for details. It corresponds directly to the so-called risk-neutral pricing methodology.*

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